The Geometry and Field Theory of Deformed Very Special Relativity

Collaborator: Zhang Lei

Xue Xun

East China Normal University

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Table of contents

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- Lorentz violation, the theoretical investigations
- The very special relativity model
- The deformed $Sim(2)$ very special relativity is a Finsler geomrtry
- The proper subgroups of Lorentz group
- The deformation Lie group
- The deformed Poincaré subgroup
- The invariant metric under deformed Poincaré subgroup
- The field theory construction

Lorentz violation, the theoretical investigations

- The possible Lorentz violation is an important theoretical question.
- The theoretical investigation and experimental examination of Lorentz symmetry have made considerable progress and attracted a lot of attentions since the mid of 1990s.
- Coleman and Glashow, boost invariance violation in the rest frame of the cosmic background radiation
- Colladay and Kostelecky standard model extension incorporating Lorentz and CPT violation

Finsler geometry realization of doubly special relativity

- large boosts naturally uncover the structure of spacetime at arbitrary small scales
- The modification of special relativity with an additional fundamental length scale, the Planck scale, is known as doubly special relativity(DSR)
- The realization of DSR can be noncommutative spacetime or the non-linear realization of Poincare group.
- deformed dispersion relation, the main feature of DSR, can also leads to Finsler type of spacetime geometry

Cohen-Glashow's very special relativity model

- at low energy scales (QED + QCD), P, C and T are individually good symmetries of nature
- Cohen and Glashow argued that the local symmetry of physics might not need to be as large as Lorentz group but its proper subgroup
- the full symmetry restores to Poincare group when discrete symmetry P, T, CP or CT enters
- The Lorentz violation is thus connected with CP violation.

The identified VSR subgroups up to isomorphism

- T(2) (2-dimensional translations) with generators $T_1 = K_x + J_y$ and $T_2 = K_y - J_x$, where **J** and **K** are the generators of rotations and boosts respectively
- E(2) (3-parameter Euclidean motion) with generators T_1, T_2 and J_z,
- HOM(2) (3-parameter orientation preserving transformations) with generators T_1, T_2 and K_z
- SIM(2) (4-parameter similitude group)with generators T_1, T_2, J_z and K_z.

The realization of VSR

- TSheikh-Jabbar et.al proved that the quantum field theory on the noncommutative Moyal plane with light-like noncommutativity possesses VSR symmetry.
- Gibbons, Gomis and Pope point out the deformed ISIM(2) admits a Finsler line element
- Zhe Chang et.al.: the isometric group of a special case of (α, β)-type Finsler space is the same with the symmetry of VSR.

The deformed $Sim(2)$ very special relativity is a Finsler geometry

- Gibbons, Gomis and Pope : duo to quantum corrections or the quantum gravity effect, ISIM(2) admits a 2-parameter family of continuous deformations, none of these give rise to noncommutative translations analogous to those of the de Sitter deformation of the Poincare group: space-time remains flat
- only a 1-parameter $DISIM_b(2)$ is physically acceptable.
- The line element invariant under $DISIM_b(2)$ is Lorentz violating and of Finsler type, $ds^2 = (\eta_{\mu\upsilon} dx^\mu dx^\upsilon)^{1-b} (n_\mu dx^\mu)^{2b}$.
- The $DISIM_b(2)$ invariant equation for spin 0 field is in general a nonlocal equation, since it involves fractional derivatives.

The proper subgroups of Lorentz group

The Lorentz Lie algebra has the following Lie sub-algebra up to isomorphism

- Lie subalgebra with a single generator
- two Lie subalgebras with two generators:
  - **span** $\{r_x, b_x\}$: $[r_x, b_x] = 0$,
  - **span** $\{r_x + b_y, b_z\}$: $[b_x + r_y, b_z] = b_x + r_y$.
- one Lie subalgebras with four generators: $[t_1, t_2] = [r_z, b_z] = 0$, $[r_z, t_1] = t_2, [r_z, t_2] = -t_1$ and $[b_z, t_1] = -t_1, [b_z, t_2] = -t_2$

- four Lie subalgebras with three generators:
  - **span** $\{r_x, r_y, r_z\}$ (the $so(3)$): $[r_x, r_y] = r_z, [r_y, r_z] = r_x, [r_z, r_x] = r_y$,
  - **span** $\{b_x, b_y, r_z\}$ (the Lorentz algebra in 2+1 dimension): $[b_x, b_y] = -r_z, [b_y, r_z] = b_x, [r_z, b_x] = b_y$.
  - **span** $\{t_1, t_2, r_z\}$ (the 2 dimensional Eudlidean algebra $e(2)$): $[t_1, t_2] = 0, [r_z, t_1] = t_2, [r_z, t_2] = -t_1$.
  - **span** $\{t_1, t_2, b_z\}$(2-dimensional orientation preserving transformations group $HOM(2)$): $[t_1, t_2] = 0, [b_z, t_1] = -t_1, [b_z, t_2] = -t_2$.

Deformation of Lie Algebra

- For a Lie algebra with commutation relations,

$$[T_i, T_j] = C_{ij}^k T_k, \tag{1}$$

- suppose the structure constants of deformed Lie algebra is of the form

$$\hat{C}_{ij}^k = C_{ij}^k + tA_{ij}^k + t^2 B_{ij}^k + ... \tag{2}$$

t :deformation parameter.

- The constrain on deformed structure constants from Jacobi identity

$$[[T_i, T_j], T_k] + [[T_j, T_k], T_i] + [[T_k, T_i], T_j] = 0 \tag{3}$$

has the form

$$\hat{C}_{l[k}^m \hat{C}_{ij]}^l = \hat{C}_{lk}^m \hat{C}_{ij}^l + \hat{C}_{li}^m \hat{C}_{jk}^l + \hat{C}_{lj}^m \hat{C}_{ki}^l = 0. \tag{4}$$

- The expansion of deformed structure constant with the power of t :

$$t\left(A^m_{l[k}C^l_{ij]}+C^m_{l[k}A^l_{ij]}\right)+ \\ t^2\left(A^m_{l[k}A^l_{ij]}+B^m_{l[k}C^l_{ij]}+C^m_{l[k}B^l_{ij]}\right)+\ldots=0. \tag{5}$$

- If there exists a family of deformed Lie algebra parametrized by a continuous variable t, there should be a group of constrained equations which arise from every power of t in the above equation, as

$$A^m_{l[k}C^l_{ij]}+C^m_{l[k}A^l_{ij]}=0, \tag{6}$$

$$A^m_{l[k}A^l_{ij]}+B^m_{l[k}C^l_{ij]}+C^m_{l[k}B^l_{ij]}=0 \tag{7}$$

and etc.

- To avoid trivial deformation : $S_\mu^v = \delta_\mu^v + t\phi_\mu^v + ... \in GL(n, \mathbb{R})$, such that $\hat{C}_{ij}^k = S_c^k C_{ab}^c \left(S^{-1}\right)_i^a \left(S^{-1}\right)_j^b$ and hence

$$A_{ij}^k = \phi_l^k C_{ij}^l - C_{lj}^k \phi_i^l - C_{il}^k \phi_j^l. \tag{8}$$

- Define λ^μ as the basis vector of the original Lie algebra (the left invariant 1-form), then $d\lambda^i = -\frac{1}{2} C_{ab}^i \lambda^a \wedge \lambda^b$[1] [7]. Define one form field $\Phi^a = \phi_b^a \lambda^b$ and 2-form field $A^a = \frac{1}{2} A_{ij}^a \lambda^i \wedge \lambda^j$ and $B^a = \frac{1}{2} B_{ij}^a \lambda^i \wedge \lambda^j$ a matrix valued 1-form field $C_a^b = \lambda^c C_{ca}^b$. So we have the covariant exterior differential operator of the present Lie algebra $D = d + C\wedge$, the formula (6) can be rewritten as

$$DA^a = 0, A^a \neq -D\Phi^a. \tag{9}$$

- The Jacobi Identity requires $D^2 = 0$, then

$$DB^a + (A \bullet A)^a = 0, \tag{10}$$

where $(A \bullet A)^a = \frac{1}{2} A^a_{b[c} A^b_{de]} \lambda^c \wedge \lambda^d \wedge \lambda^e$. The equation is solvale requires $D(A \bullet A)^a = 0$.
- If we set $A \bullet A = 0$, we find that the second order term of deformation will also satisfy (9). Then the accetable form of B^μ is the same as one of A^μ. It is enough to consider the first order deformed term only.

The Perturbative Solution of the Representation of the Deformation group Generators

- The natural representation of the deformed generators is the representation inherit from the Poincaré group's 5 dimensional natural matrix representation
- Suppose $\{T'_i = T_i + \tau G_i\}$ and $C'^k_{ij} = C^k_{ij} + tA^k_{ij}$, hence
- $C^k_{ij} T_k = [T_i, T_j]$
- $C'^k_{ij} T'_k = [T'_i, T'_j]$,
- $\tau^2 [G_i, G_j] + \tau \left([G_i, T_j] + [T_i, G_j] - C^k_{ij} G_k - tA^k_{ij} G_k \right) - tA^k_{ij} T_k = 0$, where Ts and Gs are all 5×5 matrices

- $N \times 5 \times 5 = 25N$ unknown variables for a Lie algebra with N generators, e.g. 250 unknown for Poincaré group, 200 for *ISIM* group and 175 for *IHOM* group
- In general, we can assume that $tA_{ij}^k = \tau \bar{A}_{ij}^k$,

$$\left\{ \begin{array}{l} [G_i, G_j] - \bar{A}_{ij}^k G_k = 0 \\ [G_i, T_j] + [T_i, G_j] - C_{ij}^k G_k - \bar{A}_{ij}^k T_k = 0 \end{array} \right. \tag{11}$$

- The simplest case is $t_1 A_{ij}^k = \bar{A}_{ij}^k$ and $t = t_1 \tau$. Rewrite t_1 as t

$$\left\{ \begin{array}{l} [G_i, G_j] - tA_{ij}^k G_k = 0 \\ [G_i, T_j] + [T_i, G_j] - C_{ij}^k G_k - tA_{ij}^k T_k = 0. \end{array} \right. \tag{12}$$

- There may be more than one set of solutions, which corresponding to different spacetime geometry.

Example: the deformation group of SIM

- the semi-direct product of SIM with $T(4)$, $ISIM$:

$$\begin{aligned}
&[t_1, r_z] = -t_2, [t_1, b_z] = t_1, [t_1, p_t] = [t_1, p_z] = p_x,\\
&[t_2, r_z] = t_1, [t_2, b_z] = t_2, [t_2, p_t] = [t_2, p_z] = p_y,\\
&[t_1, p_x] = p_t - p_z, [t_2, p_y] = p_t - p_z, [r_z, p_x] = p_y,\\
&[r_z, p_y] = -p_x, [b_z, p_t] = p_z, [b_z, p_z] = p_t.
\end{aligned} \tag{13}$$

- The Jacobi constrain reduces the $8 \times \frac{8\times 7}{2} = 224$ deformation parameters to 57. The simplest solution $A \bullet A = 0$ reduces further to 6 ones,

$$A^1_{1b}, A^t_{1x}, A^z_{1x}, A^t_{rt}, A^t_{bt}, A^z_{bt}, \tag{14}$$

where r, b, t, x, z represent r_z, b_z, p_t, p_x, p_z respectively.

The commutation relation for *DISIM* is

$$\begin{aligned}
&[t_1, r_z] = -t_2, [t_1, b_z] = \left(1 + A^1_{1b}\right) t_1, [t_2, r_z] = t_1, \\
&[t_1, p_t] = p_x, [t_1, p_x] = \left(1 + A^t_{1x}\right) p_t - \left(1 - A^z_{1x}\right) p_z, \\
&[t_1, p_z] = \left(1 + A^t_{1x} + A^z_{1x}\right) p_x, [t_2, b_z] = \left(1 + A^1_{1b}\right) t_2, \\
&[t_2, p_t] = p_y, [t_2, p_y] = \left(1 + A^t_{1x}\right) p_t - \left(1 - A^z_{1x}\right) p_z, \\
&[t_2, p_z] = \left(1 + A^t_{1x} + A^z_{1x}\right) p_y, [r_z, p_t] = A^t_{rt} p_t, \\
&[r_z, p_x] = p_y + A^t_{rt} p_x, [r_z, p_y] = -p_x + A^t_{rt} p_y, \\
&[r_z, p_z] = A^t_{rt} p_z, [b_z, p_t] = p_z + A^t_{bt} p_t + A^z_{bt} p_z, \\
&[b_z, p_x] = \left(A^t_{1x} + A^z_{1x} + A^t_{bt} + A^z_{bt} - A^1_{1b}\right) p_x, \\
&[b_z, p_y] = \left(A^t_{1x} + A^z_{1x} + A^t_{bt} + A^z_{bt} - A^1_{1b}\right) p_y, \\
&[b_z, p_z] = p_t + \left(2A^1_{1b} - A^z_{bt}\right) p_t \\
&+ \left(2A^t_{1x} + 2A^z_{1x} + A^t_{bt} + 2A^z_{bt} - 2A^1_{1b}\right) p_z.
\end{aligned} \tag{15}$$

- The non-triviality condition is

$$A^{t}_{rt}{}^{2} + \left(A^{t}_{1x} + A^{z}_{1x} + A^{t}_{bt} + A^{z}_{bt} - A^{1}_{1b}\right)^{2} \neq 0. \qquad (16)$$

- The simplest solution $A \bullet A = 0$ gives

$$\begin{cases} A^{z}_{1x}\left(A^{t}_{1x} + A^{z}_{1x}\right) = 0 \\ A^{z}_{bt}\left(A^{t}_{1x} + A^{z}_{1x}\right) = 0 \\ \left(A^{t}_{1x} - 2A^{1}_{1b}\right)\left(A^{t}_{1x} + A^{z}_{1x}\right) = 0 \end{cases} \qquad (17)$$

- A^{1}_{1b} -deformation inside of the original *sim* .We thus can classify *DISIM* into two classes.
- There are many subcases.

The example case, $A^t_{bt} = 0$

In the example case: $A^t_{bt} = 0$. Denoting $A_1 = A^t_{rt}$ and $A_2 = A^t_{bt}$, the representation matrices of the deformed generators are

$$
\begin{aligned}
r_z &= \begin{pmatrix} A_1 & & & & \\ & A_1 & -1 & & \\ & 1 & A_1 & & \\ & & & A_1 & \\ & & & & 0 \end{pmatrix}, \\
b_z &= \begin{pmatrix} A_2 & & & 1 & \\ & A_2 & & & \\ & & A_2 & & \\ 1 & & & A_2 & \\ & & & & 0 \end{pmatrix},
\end{aligned}
\tag{18}
$$

The corresponding single parameter group elements are

$$
\begin{aligned}
R_z(\theta) &= \begin{pmatrix} e^{\theta A_1} & & & & \\ & e^{\theta A_1}\cos\theta & -e^{\theta A_1}\sin\theta & & \\ & e^{\theta A_1}\sin\theta & e^{\theta A_1}\cos\theta & & \\ & & & e^{\theta A_1} & \\ & & & & 1 \end{pmatrix}, \\
B_z(\theta) &= \begin{pmatrix} e^{\theta A_2}\cosh\theta & & & e^{\theta A_2}\sinh\theta & \\ & e^{\theta A_2} & & & \\ & & e^{\theta A_2} & & \\ e^{\theta A_2}\sinh\theta & & & e^{\theta A_2}\cosh\theta & \\ & & & & 1 \end{pmatrix},
\end{aligned}
\tag{19}
$$

Note: $R_z(\theta)$ is a rotation followed by a dilatation $e^{\theta A_1}$.
$R_z(2\pi) = e^{2\pi A_1}$ is a pure dilatation when $A_1 \neq 0$. To keep $R_z(\theta)$ as a reasonable local rotation operation, one demands $A_1 = 0$.
Denoted A_2 by b the deformed boost operation :

$$B_z(\theta) = e^{b\theta} \begin{pmatrix} \cosh\theta & & & \sinh\theta \\ & 1 & & \\ & & 1 & \\ \sinh\theta & & & \cosh\theta \end{pmatrix}, \qquad (20)$$

an ordinary boost followed by a dilatation.

Summary of deform group

Summary: the deformation of semi-direct product of all of three and four generators Lorentz subgroups with $T(4)$ and their natural representations

Table : The Deformation of semi-direct product Poincaré subgroups.

subgroup	class	subclass	natural rep.	remark
Poincaré	de Sitter	de Sitter	1	the isometry group of maximal symmetric space of 4-spacetime
ISIM	*DISIM* (*SIM* undeformed)	*DISIM*	1	lots of equivalent deformation corresponding to generators redefinition and additional accompanied dilatation for rotation and boost operation
	*XDISIM*1 (*SIM* deformed)	*XDISIM*1	1	lots of equivalent deformation corresponding to generators redefinition and additional accompanied dilatation for rotation and boost operation
	*XDISIM*2 (*SIM* deformed)	*XDISIM*2	1	additional accompanied dilatation for rotation operation additional accompanied dilatation for boost operation

IHOM	*DIHOM*1 (*WDISIM*)	*DIHOM*1 (*WDISIM*)	1	lots of equivalent representations corresponding to generators redefinition, additional accompanied dilatation for boost operation same structure as the corresponding part of *DISIM*
	*DIHOM*2 (*DIHOM*)	*DIHOM*2 (*DIHOM*)	1	no natural representations inherited from Poincaré group additional accompanied dilatation for boost operation
TE(2)	*DTE*1	*DTE*1	1	additional accompanied dilatation for rotation operation rotation operation not only in *xy* plane but also in rotated *tz* plane
	*DTE*2	*DTE*2*a*	2	translations are entangled with t_1 and t_2 operations
		*DTE*2*b*	0	no natural representation inherited from Poincaré group
	*DTE*3	*DTE*3*a*	2	translations are entangled with t_1 and t_2 operations
		*DTE*3*b*	1	translations are entangled with t_1 and t_2 operations rotation operation not only in *xy* plane but also in rotated *tz* plane
ISO(3)	*DISO*(3)1	*DISO*(3)1	1	inequivalent representation corresponding to different sign of deform parameter, only translations operations deformed
	DISO(3)2	*DISO*(3)2	3	three inequivalent representations only translations operations deformed
ISO(2, 1)	*DISO*(2, 1)1	*DISO*(2, 1)1	1	inequivalent representation corresponding to different sign of deform parameter, only translations operations deformed
	DISO(2, 1)2	*DISO*(2, 1)2	2	two inequivalent representations only translations operations deformed

Minkowski-Finsler Manifold

- The geometry with deformed Poincaré subgroup symmetry is usually a Finsler Geometry.
- In Finsler geometry, Minkowski-Finsler manifold is a class of flat manifolds whose Finsler norm does not change with the coordinate on the base manifold and hence a function of the coordinate of the vector space, $F = F(y^\alpha)$.
- We are seeking the Minkowski-Finsler type of geometry with deformed Poincaré subgroup symmetry.

The Invariant Metric

- Without losing generality

$$F^2 = \prod_{i=1}^{M} F_i. \tag{21}$$

$$F_i = M_i^{E_i}, \tag{22}$$

- where E_i is constant and M_i satisfies

$$M_i\left(y^\mu\right) = G_{\mu_1\mu_2\ldots\mu_{p_i}} \prod_{j=1}^{p_i} y^{\mu_j}. \tag{23}$$

- The $G_{\mu_1\mu_2\ldots\mu_{p_i}}$ is constant tensor. For F^2 is a degree 2 homogenous function of y_μ, we have

$$\sum_{i=1}^{M} p_i E_i = 2. \tag{24}$$

- Suppose T_a is a single parameter group element we can demand

$$M_i\left(T_a\left(y^\mu\right)\right)=A_{ia}M_i\left(y^\mu\right). \tag{25}$$

- For F^2 is invariant under the action of T_a, we have

$$\prod_{i=1}^{M}A_{ia}^{E_i}=1, \tag{26}$$

Summary of The Invariant Metric

Table : The Finsler spacetime with symmetry group of Poincaré subgroups and deformed Poincaré subgroups

symmetric group	conformal covariant tensor	conformal factor
	the invariant metric and additional remark	
de Sitter	no Minkowski-Finsler type of invariant metric	
Poincaré	$G_{\mu\upsilon} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	1
	$F^2 = G_{\mu\upsilon} y^\mu y^\upsilon$	

DISIM	$N_\mu = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}^{\mathbf{T}}$	$B_z(\theta) : e^{(1+A_2)\theta}$
	$G_{\mu\upsilon}$	$B_z(\theta) : e^{2A_1\theta}$
	$F^2 = (G_{\mu\upsilon} y^\mu y^\upsilon)^{1+A_2} (N_\mu y^\mu)^{-2A_2}$	
XDISIM1	N_μ	$B_z(\theta) : e^{(1+A_3)\theta}$
	$G_{\mu\upsilon}$	$B_z(\theta) : e^{2(A_3-A_1)\theta}$
	$F^2 = (G_{\mu\upsilon} y^\mu y^\upsilon)^{\frac{1+A_3}{1+A_1}} (N_\mu y^\mu)^{-2\frac{A_3+A_1}{1+A_1}}$ no invariant metric incase of $A_1 = -1$	

$XDISIM2$	N_μ	$B_z(\theta) : e^{(1+A_3)\theta}$
	$H_{(M,N)\mu\upsilon}$ where $M = -\frac{1+A_3}{1+A_1}, N = \frac{A_1-A_3}{1+A_1}$	$B_z(\theta) : e^{2(A_3-A_1)\theta}$
	$F^2 = \left(H_{(M,N)\mu\upsilon}y^\mu y^\upsilon\right)^{\frac{1+A_3}{1+A_1}} \left(N_\mu y^\mu\right)^{-2\frac{A_3-A_1}{1+A_1}}$ a $t-z$ plane non-orthogonal linear transformation is made relative to $DISIM$	
$ISIM$	N_μ	$B_z(\theta) : e^\theta$
	$G_{\mu\upsilon}$	invariant
	$F^2 = G_{\mu\upsilon}y^\mu y^\upsilon$	

$DIHOM$	no invariant metric function	
$WDIHOM$	the same as $DISIM$	
$IHOM$	the same as $ISIM$	
$DTE1$	no invariant metric function	
$DTE2a1$	N_μ	invariant
	$G_{\mu\upsilon}$	$P_t\left(\theta\right), P_z\left(\theta\right) : e^{A_2\theta}$
	$F = N_\mu y^\mu$ the relation between two deform parameters: $A_1 = A_2^2/4$	

$DTE2a2$	N_μ	$P_t(\theta), P_z(\theta) : e^{(2\lambda - A_2)\theta}$
	$G_{\mu\upsilon}$	$P_t(\theta), P_z(\theta) : e^{2\lambda\theta}$
	$F^2 = (G_{\mu\upsilon} y^\mu y^\upsilon)^{\frac{A_2 - 2\lambda}{A_2 - \lambda}} (N_\mu y^\mu)^{\frac{2\lambda}{A_2 - \lambda}}$ deform parameters satisfy: $\lambda^2 - A_2\lambda + A_1 = 0$ and $\lambda \neq A_2$	
$DTE2b$	no invariant metric function	
$DTE3a$	the same as $DTE2a$	

<table>
<tr><td rowspan="3">$DTE3b$</td><td>N_μ</td><td>invariant</td></tr>
<tr><td>$H_{(a,b)\mu\upsilon} = \begin{pmatrix} a & & & a+b \\ & b & & \\ & & b & \\ a+b & & & a+2b \end{pmatrix}$</td><td>invariant</td></tr>
<tr><td colspan="2">$F^2 = \prod_{a,b} \left(H_{(a,b)\mu\upsilon} y^\mu y^\upsilon \right)^{D_{a,b}}$
the constrain condition: $\sum_{a,b} D_{a,b} = 1$</td></tr>
</table>

$TE(2)$	the same as $DTE3b$ and hence denote $DTE3b$ by $TE(2)$
$DISO(3)1$	no invariant metric
$DISO(3)2$	no invariant metric
$DISO(2,1)1$	no invariant metric
$DISO(2,1)2$	no invariant metric

$ISO(3)$	$T_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T$	invariant
	$G_{(a,b)\mu\upsilon} = \begin{pmatrix} a & & & \\ & b & & \\ & & b & \\ & & & b \end{pmatrix}$	invariant
	$F^2 = (T_\mu y^\mu)^A \prod_{a,b} \left(G_{(a,b)\mu\upsilon} y^\mu y^\upsilon \right)^{B_{a,b}}$ the constrain condition: $A + 2\sum_{a,b} B_{a,b} = 2$	

$ISO(2,1)$	$X_\mu = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^T$	invariant
	$\tilde{G}_{(a,b)\mu\upsilon} = \begin{pmatrix} a & & & \\ & b & & \\ & & -a & \\ & & & -a \end{pmatrix}$	invariant
	$F^2 = (X_\mu y^\mu)^A \prod_{a,b} \left(\tilde{G}_{(a,b)\mu\upsilon} y^\mu y^\upsilon\right)^{B_{a,b}}$ the constrain condition: $A + 2\sum_{a,b} B_{a,b} = 2$	

- The invariant metric function for deformed Poincaré subgroup can be written as

$$F^2 = (A_\mu y^\mu)^{2-2\sum\limits_{a,b} D_{a,b}} \prod_{a,b} \left(B_{(a,b)\mu\upsilon} y^\mu y^\upsilon\right)^{D_{a,b}}.$$

  where A_μ can be N_μ, T_μ and X_μ while $B_{(a,b)\mu\upsilon}$ can take $\tilde{G}_{(a,b)\mu\upsilon}$, $B_{(a,b)\mu\upsilon}$ and $H_{(a,b)\mu\upsilon}$. For different groups, the metric usually are: $F^2 = G_{\mu\upsilon} y^\mu y^\upsilon$, $(N_\mu y^\mu)^2$ or $(G_{\mu\upsilon} y^\mu y^\upsilon)^{1-A} (N_\mu y^\mu)^{2A}$.
- metric function can be constructed by adding different parts, e.g. TE(2) can have such form metric function

$$F = A\sqrt{G_{\mu\upsilon} y^\mu y^\upsilon + (N_\mu y^\mu)^2} + B\sqrt{G_{\mu\upsilon} y^\mu y^\upsilon} + C N_\mu y^\mu. \quad (27)$$

- Among undeformed groups, only the *ISIM* group invariant metric function is the Minkowskian while the metric TE(2), ISO(3) and ISO(2,1) invariant are all of the deformed form.
- the existence of invariant metric function automatically excludes the additional accompanied scale transformation for rotation operation, i.e. it is a requirement of geometry that the rotation operation is unchanged even in a Lorentz violation theory

More Forms of Metric Function

- Metric function can have plenty structure,
- If there exist some scalar function $\phi\left(y^{\mu}\right)$ which is the zero degree homogenous function of y^{μ} and invariant, the product of ϕ and the metric function is still an invariant metric function.

Table : invariant zero degree functions of deformed Poincaré subgroup

symmetric group	invariant zero degree homogenous function ϕ
DISIM	$\phi=1$
XDISIM1	$\phi=1$
XDISIM2	$\phi=1$
ISIM	$\phi=1$
DTE2a1	$\phi=1$
DTE2a2	$\phi=1$
DTE3b	$\phi=1$
TE(2)	$\phi_{a,b;c,d} = \frac{H_{(a,b)\mu\upsilon}y^\mu y^\upsilon}{H_{(c,d)\mu\upsilon}y^\mu y^\upsilon}$
ISO(3)	$\phi_{a,b} = \frac{(T_\mu y^\mu)^2}{G_{(a,b)\mu\upsilon}y^\mu y^\upsilon}$ and $\phi_{a,b;c,d} = \frac{G_{(a,b)\mu\upsilon}y^\mu y^\upsilon}{G_{(c,d)\mu\upsilon}y^\mu y^\upsilon}$
ISO(2,1)	$\phi_{a,b} = \frac{(X_\mu y^\mu)^2}{\tilde{G}_{(a,b)\mu\upsilon}y^\mu y^\upsilon}$ and $\phi_{a,b;c,d} = \frac{\tilde{G}_{(a,b)\mu\upsilon}y^\mu y^\upsilon}{\tilde{G}_{(c,d)\mu\upsilon}y^\mu y^\upsilon}$

- For DTE3b, TE(2), ISO(3), ISO(2,1), the invariant metric function can take the form of

DTE3b:	$F^2 = (G_{\mu\upsilon} y^\mu y^\upsilon)^{1-A} (N_\mu y^\mu)^{2A} S\left(\phi_{\mathrm{DTE3b}}\right)$
TE (2):	$F^2 = \prod_{a,b} \left(H_{(a,b)\mu\upsilon} y^\mu y^\upsilon\right)^{D_{a,b}} S\left(\phi_{\mathrm{TE}(2)a,b;c,d}\right)$
ISO (3):	$F^2 = (T_\mu y^\mu)^A \prod_{a,b} \left(G_{(a,b)\mu\upsilon} y^\mu y^\upsilon\right)^{B_{a,b}} S\left(\phi_{\mathrm{ISO}(3)a,b}, \phi_{\mathrm{ISO}(3)a,b;c,d}\right)$
ISO (2, 1):	$F^2 = (X_\mu y^\mu)^A \prod_{a,b} \left(\tilde{G}_{(a,b)\mu\upsilon} y^\mu y^\upsilon\right)^{B_{a,b}} S\left(\phi_{\mathrm{ISO}(2,1)a,b}, \phi_{\mathrm{ISO}(2,1)a,b;c,d}\right)$

- where S is an arbitrary function.

The Physics of Deformed Very Special Relativity

- the action for point particle
- In Finsler spacetime, the action for free point particle has the form of

$$S = \int_{\tau_1}^{\tau_2} mF\left(x^\mu, V^\mu\right) d\tau = \int_{t_1}^{t_2} \frac{mF\left(x^\mu, V^\mu\right)}{V^t} dt$$

  where $V^\mu = \frac{dx^\mu}{d\tau}$
- The lagrangian is

$$L = \frac{mF\left(x^\mu, V^\mu\right)}{V^t} = mF\left(x^\mu; v^\mu\right)$$

  where $v^\mu = \frac{V^\mu}{V^t}$

- in DISIM spacetime, the lagrangian for point particle is

$$L = m(G_{\mu\upsilon}v^{\mu}v^{\upsilon})^{\frac{1-A}{2}}(N_{\mu}v^{\mu})^{A}$$

- the conjugate momentum is

$$p_{\mu} = \frac{\partial L}{\partial v^{\mu}} = L\left[(1-A)\,G_{\mu\upsilon}v^{\upsilon}(G_{\mu\upsilon}v^{\mu}v^{\upsilon})^{-1} + AN_{\mu}(N_{\mu}v^{\mu})^{-1}\right]$$

- The momentum can be decomposed into kinematic part and the interacting part

$$\begin{cases} p_{\mu} = k_{\mu} + f_{\mu} \\ k_{\mu} = (1-A)\,L(G_{\mu\upsilon}v^{\mu}v^{\upsilon})^{-1}G_{\mu\upsilon}v^{\upsilon} \\ f_{\mu} = AL(N_{\mu}v^{\mu})^{-1}N_{\mu} \end{cases}$$

- dispersion relation

$$\left(G^{\mu\upsilon}p_{\mu}p_{\upsilon}\right)^{1+A}\left(N^{\mu}p_{\mu}\right)^{-2A}=m^2\left(1-A^2\right)^{1+A}(1-A)^{-2A}$$

- similar relation for the kinematic momentum

$$\begin{cases} G^{\mu\upsilon}k_{\mu}k_{\upsilon}=(1-A)^2L^2\left(G_{\mu\upsilon}v^{\mu}v^{\upsilon}\right)^{-1} \\ N^{\mu}k_{\mu}=(1-A)\,L\left(G_{\mu\upsilon}v^{\mu}v^{\upsilon}\right)^{-1}N_{\mu}v^{\upsilon} \\ \left(G^{\mu\upsilon}k_{\mu}k_{\upsilon}\right)^{1+A}\left(N^{\mu}k_{\mu}\right)^{-2A}=(1-A)^2m^2 \end{cases}$$

The Action for DISIM

- For DISIM group, the boost operation has additional dilatation
- dilatation is commutative with all the group operation, one can add an additional conformal factor to the original representation of the group to construct the new representation
- scalar, spinor and vector fields have additional conformal factor

- For DISIM group, from the massless dispersion relation of point particle, one can get the kinematic part of scalar lagrangian

$$L_m = C(\partial_\mu \phi^* \partial^\mu \phi)^r (N^\mu \phi^* \partial_\mu \phi - N^\mu \phi \partial_\mu \phi^*)^s$$

- $B_z(\theta) : \phi \to e^{\frac{s-4A}{2(r+s)}\theta} \phi$
- the mass term for scalar field can be introduced by

$$L_M = D(\phi^* \phi)^a (N^\mu \phi^* \partial_\mu \phi - N^\mu \phi \partial_\mu \phi^*)^b$$

where $b = \frac{4Ar - 4Aa + 4As + as}{r + 4A}$ and a, D to be determined.

- The dispersion relation for plane wave solution is

$$(p_\mu p^\mu)^r (-2iN^\mu p_\mu)^{s-b} = -\frac{D(a+b)}{C(r+s)}$$

- comparing with point particle's dispersion relation, one get

$$\begin{cases} b = s + \frac{2Ar}{1+A} \\ s = 2A\left(2 - \frac{r}{1+A}\frac{r+4A}{r-a}\right) \\ -\frac{D(a+b)}{C(r+s)} = m^{\frac{2r}{1+A}}\left(1-A^2\right)^r\left[-2i\left(1-A\right)\right]^{-\frac{2Ar}{1+A}} \end{cases}$$

The Action for Fields

two key point to construct the action for fields,

- the action is invariant under the group action
- the plane wave solution of fields satisfy the dispersion relation for point particle

- one get finally

$$\begin{cases} a = 1 + n\sqrt{A}, b = s + 2A, s = 4A + 2\sqrt{A}\frac{1+5A}{n-\sqrt{A}}, D = \frac{a+2A+s}{1+A+s}m^2 \\ j_\mu = \frac{i}{2(1-A)}\left(\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right), C = \left(1 - A^2\right)^{-(1+A)}, \end{cases}$$

- The lagrangian is

$$L = (N^\mu j_\mu)^{s+2A}\left[\left(1 - A^2\right)^{-(1+A)}(\partial_\mu\phi^*\partial^\mu\phi)^{1+A}(N^\mu j_\mu)^{-2A} - \frac{a+2A+s}{1+A+s}m^2(\phi^*\phi)^{1+n\sqrt{A}}\right]$$

- expansion in the deformation parameter

$$L = \partial^\mu\phi^*\partial_\mu\phi - \left(1 + n\sqrt{A}\right)m^2\phi^*\phi - \frac{2\sqrt{A}}{n}\left[(\partial^\mu\phi^*\partial_\mu\phi) - m^2\phi^*\phi\right]\ln\left(N^\mu\tilde{j}_\mu\right)$$

- similar result for the spinor field of DISIM

$$L = C\left(\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^\mu\psi\right)^r\left(N_\mu\bar{\psi}\gamma^\mu\psi\right)^s \\ +D\left(\bar{\psi}\psi\right)^a\left(N_\mu\bar{\psi}\gamma^\mu\psi\right)^b$$

where $B_z(\theta) : \psi \to e^{\frac{s-4A}{2(r+s)}\theta}\psi$ and $b = \frac{4A(r+s)+a(s-4A)}{r+4A}$

- the plane wave solution of fields give

$$\begin{cases} b = s + A\frac{2r-a}{1+A}, s = A\frac{r+4A}{r-a}\left(\frac{4r-4a}{r+4A} - \frac{2r-a}{1+A}\right) \\ \left[-\frac{D(a+b)}{C(r+s)}\right]^{\frac{2(1+A)}{2r-a}}(-2i)^{-\frac{2r(1+A)}{2r-a}} = m^2\left(1-A^2\right)^{1+A}(1-A)^{-2A} \end{cases}$$

- The action can be written finally

$$L = \left(N_\mu \bar{\psi}\gamma^\mu\psi\right)^s \left\{ \left[\frac{i}{2}\left(\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^\mu\psi\right)\right]^{1+n\sqrt{A}} \right.$$
$$\left. - M\left(\bar{\psi}\psi\right)^{1+2n\sqrt{A}}\left(\frac{N_\mu\bar{\psi}\gamma^\mu\psi}{\bar{\psi}\psi}\right)^A \right\}$$

where $s = \sqrt{A}\left(4 - \frac{1+n\sqrt{A}+4A}{\sqrt{A}-n}\right)$ and
$M = m\frac{1+n\sqrt{A}+s}{1+2n\sqrt{A}+s}\left(1-A^2\right)^{\frac{1+A}{2}}(1-A)^{-A}$

- the perturbative expansion is

$$\begin{aligned} L &= \frac{i}{2}\left(\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^\mu\psi\right) - \left(1 - n\sqrt{A}\right) m\bar{\psi}\psi \\ &+ \left(4 + \frac{1}{n}\right)\sqrt{A}\left[\frac{i}{2}\left(\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^\mu\psi\right) - m\bar{\psi}\psi\right]\ln\left(N_\mu\bar{\psi}\gamma^\mu\psi\right) \\ &+ n\sqrt{A}\left[\frac{i}{2}\left(\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^\mu\psi\right)\right]\ln\left[\frac{i}{2}\left(\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^\mu\psi\right)\right] \\ &- 2n\sqrt{A}m\left(\bar{\psi}\psi\right)\ln\left(\bar{\psi}\psi\right) \end{aligned}$$

- similar method can get the action for the vector field

$$L = (N^\mu A^\upsilon F_{\upsilon\mu})^{s+2A}\left[\left(\frac{F_{\mu\upsilon}F^{\mu\upsilon}}{4(1-A^2)}\right)^{1+A}(N^\mu A^\upsilon F_{\upsilon\mu})^{-2A} - \frac{1}{2}M^2(A^\mu A_\mu)^{1+n\sqrt{A}}\right]$$

where $s = 2\sqrt{A}\frac{1+2n\sqrt{A}+3A}{n-\sqrt{A}}$ and $M = \sqrt{\frac{1+A}{1+n\sqrt{A}}}(1-A)^{-A}m$

- the action for massless vector field in TE(2) spacetime is

$$L = \frac{1}{4}F_{\mu\upsilon}F^{\mu\upsilon} - \frac{g^2}{2}(N^\mu A^\upsilon F_{\upsilon\mu})^2$$

- the action for massless vector field in DISIM spacetime is

$$L = \left(\frac{F_{\mu\upsilon}F^{\mu\upsilon}}{4(1-A^2)}\right)(N^\mu A^\upsilon F_{\upsilon\mu})^{4A}$$

- coupling between scalar and gauge field

$$L = (N^\mu j_\mu)^{s+2A}\left[\left(1-A^2\right)^{-(1+A)}(D_\mu\phi^* D^\mu\phi)^{1+A}(N^\mu j_\mu)^{-2A}\right.$$
$$\left.-\tfrac{a+2A+s}{1+A+s}m^2(\phi^*\phi)^{1+n\sqrt{A}}\right]+\left(\tfrac{F_{\mu\upsilon}F^{\mu\upsilon}}{4(1-A^2)}\right)(N^\mu A^\upsilon F_{\upsilon\mu})^{4A}$$

- coupling between spinor and gauge field

$$L = \left(N_\mu\bar{\psi}\gamma^\mu\psi\right)^s\left\{\left[\tfrac{i}{2}\left(\bar{\psi}\gamma^\mu\partial_\mu\psi-\partial_\mu\bar{\psi}\gamma^\mu\psi\right)\right]^{1+n\sqrt{A}}\right.$$
$$\left.-M\left(\bar{\psi}\psi\right)^{1+2n\sqrt{A}}\left(\tfrac{N_\mu\bar{\psi}\gamma^\mu\psi}{\bar{\psi}\psi}\right)^A\right\}+\left(\tfrac{F_{\mu\upsilon}F^{\mu\upsilon}}{4(1-A^2)}\right)(N^\mu A^\upsilon F_{\upsilon\mu})^{4A}$$

- the action for electromagnetic coupling of point particle is

$$S = \int_{\tau_1}^{\tau_2}\left[mF\left(V^\mu\right)+eV^\mu A_\mu\right]d\tau$$
$$+\int\left(\tfrac{F_{\mu\upsilon}F^{\mu\upsilon}}{4(1-A^2)}\right)(N^\mu A^\upsilon F_{\upsilon\mu})^{4A}dV$$

The field theory of deformed very special relativity

- fields exhibit rescale effect in some specific Finsler spacetime
- Field can get an effective mass in some ISO(3) spacetime
- the effective mass depends on direction in TE(2) spacetime-the anisotropy

Conclusion

- obtained various of deformed Poincaré subgroups and their natural representations
- obtained the spacetime metric function corresponding to various of semi-direct product Poincaré subgroups and their deform partner
- obtained the field theory in various of spacetime with semi-direct product Poincaré subgroups and their deform partner symmetry

Thanks for your attention!