Effect of Generalized Uncertainty Principle on Quantum Tunneling of Black Hole

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Motivation

- By invoking Generalized Uncertainty Principle (GUP) and some heuristic arguments, it has been shown that the BH temperature should be corrected as the black hole mass approaches the Planck-scale.
- However, a direct way to calculate the corrected BH thermodynamics is still lacking.
- Our goal is to derive the quantum gravity correction to black hole temperature due to generalized uncertainty principle (GUP) in a systematic approach.

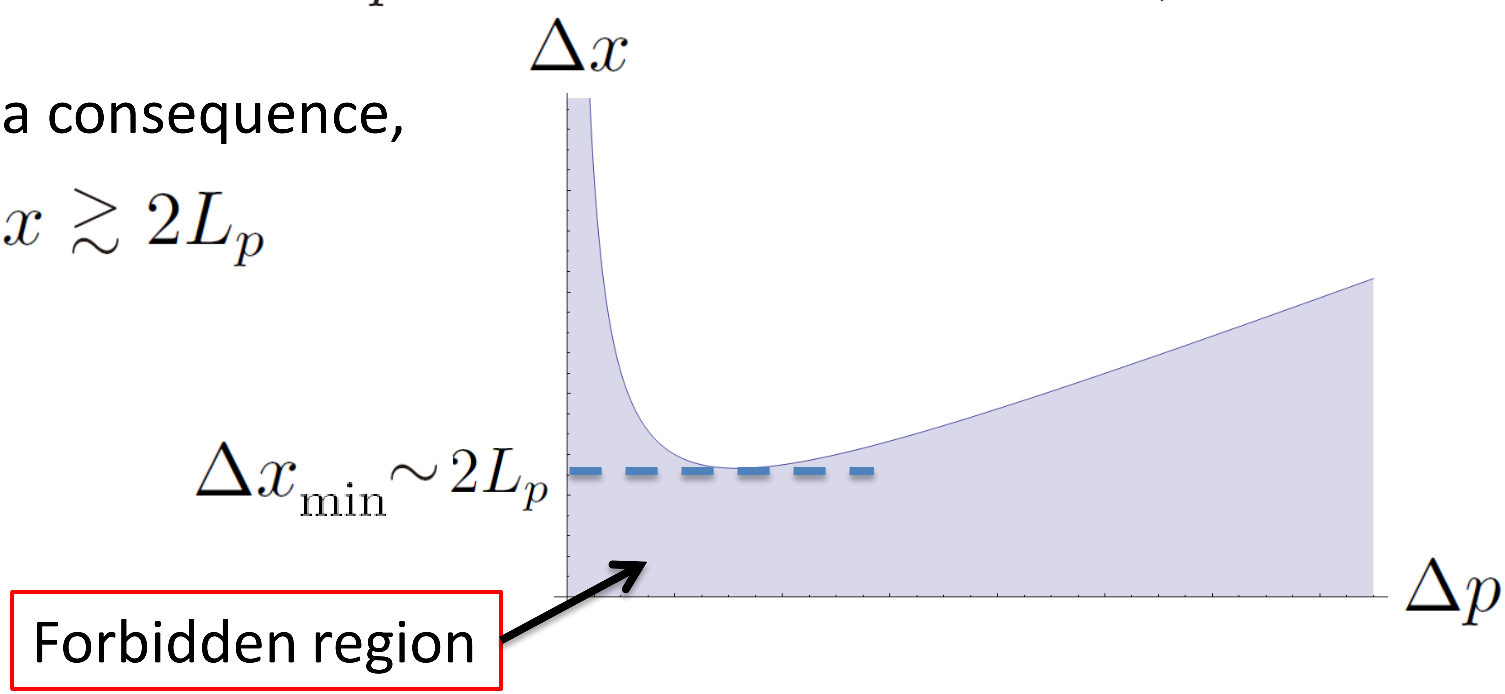
Generalized Uncertainty Principle

Studies including string theory and quantum gravity suggest the existence of a minimal observable length, and actually leads to a generalized uncertainty principle:

(GUP): $\Delta x \gtrsim \frac{\hbar}{\Delta p} + L_p^2 \frac{\Delta p}{\hbar}$,where $L_p \equiv \sqrt{\frac{G\hbar}{c^3}}$

As a consequence,

$$\Delta x \gtrsim 2L_p$$



GUP and BH Thermodynamics

Based on GUP and some heuristic arguments, the corrected BH temperature and entropy have been studied.[1]

$$T_{\mathrm{GUP}} = \frac{Mc^2}{4\pi k_B}\left[1-\sqrt{1-\frac{{M_p}^2}{M^2}}\right] = \frac{c^2{M_P}^2}{8\pi k_B M}\left[1+\frac{{M_P}^2}{4M^2}+\frac{{M_P}^4}{8M^4}+\ldots\right]$$

$$S_{\mathrm{GUP}} = 2\pi k_B\left\{\frac{M^2}{{M_p}^2}\left(1-\frac{{M_p}^2}{M^2}+\sqrt{1-\frac{{M_p}^2}{M^2}}\right) - \mathrm{Log}\left[\frac{M}{M_p}\left(1+\sqrt{1-\frac{{M_p}^2}{M^2}}\right)\right]\right\}$$

$$= 4\pi k_B\frac{M^2}{{M_p}^2} - \pi k_B \mathrm{Log}\left(\frac{M^2}{{M_p}^2}\right) + \mathrm{const.} + \ldots$$

$$= k_B\frac{A}{4{L_p}^2} - \pi k_B \mathrm{Log}\left(\frac{A}{{L_p}^2}\right) + \mathrm{const.} + \ldots\ ,$$

[1] R. J. Adler, P. Chen, and D. I. Santiago, Gen. (2001).

Kempf's approach: small quadratic corrections to the CCR

Kempf [2] suggested to recontruct QM from a modified canonical commutation relation (CCR) which lead to GUP naturally, for one dimention: $[\mathbf{x}, \mathbf{p}] = i\hbar(1 + \beta \mathbf{p}^2)$

$$\Delta x \Delta p \geq \frac{\hbar}{2}(1 + \beta(\Delta p)^2 + \beta \langle \mathbf{p} \rangle^2) \Rightarrow \Delta x_0 = \hbar\sqrt{\beta}$$

Therefore the representation on momentum space now become:

$$\begin{cases} \mathbf{p}.\psi(p) = p\psi(p) \\ \mathbf{x}.\psi(p) = i\hbar(1 + \beta p^2)\partial_p \psi(p) \end{cases}$$

The Modified canonical commutation relations for n-dimention:

$$[X^i, P^j] = -i\hbar[(1 + \beta \boldsymbol{P}^2)g^{ij} - \beta' P^i P^j],$$

$$[X^i, X^j] = i\hbar \frac{2\beta - \beta' + (2\beta + \beta')\beta \boldsymbol{P}^2}{1 + \beta \boldsymbol{P}^2}(P^i X^j - P^j X^i),$$

$$[P^i, P^j] = 0,$$

[2] A. Kempf, G. Mangano and R. B. Mann: Phys. Rev. D 52 (1995)

Kempf's Theory Violate Lorentz Covariance

Inspired by Kempf's theory (1995), many authors reinvestigated all kinds of (non-relativistic) QM problem.

Kempf added a small quadratic corrections to the CCR, and thus lead to a GUP of the required form, but **the resulting algebra is not Lorentz covariant.**

In 2006, Quesne and Tkachuk[3] introduced a generalization of Kempf's modified CCR to a Lorentz-covariant form.

[3] C. Quesne and V. M. Tkachuk, J. Phys. A: Math. Gen. 39, 10909 (2006).

Lorentz-Covariant Deformed Algebra Introduced by Quesne and Tkachuk

Kempf's algebra :

$$\begin{cases} [X^i, P^j] = -i\hbar[(1+\beta \boldsymbol{P}^2)g^{ij} - \beta' P^i P^j], \\ [X^i, X^j] = i\hbar \frac{2\beta - \beta' + (2\beta + \beta')\beta \boldsymbol{P}^2}{1+\beta \boldsymbol{P}^2}(P^i X^j - P^j X^i), \\ [P^i, P^j] = 0, \end{cases}$$

In momentum representation, the position and momentum operators are represented by:

$$X^i = (1+\beta \boldsymbol{p}^2)x^i + \beta' p^i (\boldsymbol{p}\cdot\boldsymbol{x}) + i\hbar\gamma p^i, \qquad P^i = p^i, \quad \text{where } x^i = i\hbar\partial/\partial p^i$$

Lorentz-covariant algebra generalizing that of kempf:

$$\begin{cases} [X^\mu, P^\nu] = -i\hbar[(1-\beta P_\rho P^\rho)g^{\mu\nu} - \beta' P^\mu P^\nu], \\ [X^\mu, X^\nu] = i\hbar \frac{2\beta - \beta' - (2\beta + \beta')\beta P_\rho P^\rho}{1-\beta P_\rho P^\rho}(P^\mu X^\nu - P^\nu X^\mu), \\ [P^\mu, P^\nu] = 0, \end{cases}$$

$$X^\mu = (1-\beta p_\nu p^\nu)x^\mu - \beta' p^\mu p_\nu x^\nu + i\hbar\gamma p^\mu, \qquad P^\mu = p^\mu,$$

Lorentz-Covariant Deformed Algebra

Now the Generalized uncertainty principle (in D dimention) becomes:

$$\Delta X^i \Delta P^i \geq \frac{\hbar}{2} \left| 1 - \beta \left\{ \langle (P^0)^2 \rangle - \sum_{j=1}^{D} \left[(\Delta P^j)^2 + \langle P^j \rangle^2 \right] \right\} + \beta' \left[(\Delta P^i)^2 + \langle P^i \rangle^2 \right] \right|$$

$$(\Delta X)_0 = (\Delta X^i)_0 = \hbar \sqrt{(D\beta + \beta') \left[1 - \beta \langle (P^0)^2 \rangle \right]}$$

It should be note that **the uncertainty of position is frame-dependent** now.

The Klein-Gordon Equation Derived from Lorentz-Covariant Deformed Algebra

The Klein-Gordon equation in a (3+1)-dimensional space-time described by Quesne-Tkachuk Lorentz-covariant deformed algebra is studied[4] .
(in the case where β'= 2β, and use the approximation up to first order in β)

In such a first order approximation, the Lorentz-covariant deformed algebra reads:

$$[X^\mu, P^\nu] = -i\hbar(g^{\mu\nu}(1 - \beta P_\rho P^\rho) - 2\beta P^\mu P^\nu),$$
$$[X^\mu, X^\nu] = 0,$$
$$[P^\mu, P^\nu] = 0.$$

[4] Moayedi, Setare, Moayeri.Int.J.Theor.Phys.49 (2010)

The Klein-Gordon Equation Derived from Lorentz-Covariant Deformed Algebra

Now the following representations satisfy the above algebra:

$$P^\mu = (1 - \beta p^2)p^\mu, \quad X^\mu = x^\mu,$$

Where $p^2 = p_\alpha p^\alpha$ and $x^\mu, p^\mu = i\hbar\frac{\partial}{\partial x_\mu} = i\hbar\partial^\mu$ are position and momentum operators in ordinary relativistic quantum mechanics, and

$$p_\mu p^\mu \Phi - m^2c^2\Phi = 0$$

$$\Rightarrow (1 - \beta p^2)p_\mu(1 - \beta p^2)p^\mu\Phi - m^2c^2\Phi = 0$$

$$\Rightarrow \Box\Phi + 2\beta\hbar^2\Box\Box\Phi + (\frac{mc}{\hbar})^2\Phi = 0$$

(up to first order in β)

Hawking Radiation as A Tunneling Effect

Schwarzschild metric: $ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2$, $f(r) = 1 - r_H/r,\ r_H = 2M$

Klein-Gordon equation: $-\frac{\hbar^2}{\sqrt{-g}}\partial_\mu[g^{\mu\nu}\sqrt{-g}\partial_\nu]\phi = 0$

For radial trajectories only the (r – t) sector of the metric is important. Therefore under this metric the Klein-Gordon equation reduces to :

$$-\frac{1}{\sqrt{f(r)g(r)}}\partial_t^2\phi + \frac{1}{2}\Big(f'(r)\sqrt{\frac{g(r)}{f(r)}} + g'(r)\sqrt{\frac{f(r)}{g(r)}}\Big)\partial_r\phi + \sqrt{f(r)g(r)}\partial_r^2\phi = 0$$

The semiclassical wave function satisfying the above equation is obtained by making the standard **ansatz** for ϕ which is

$$\phi(r,t) = \exp\Big[-\frac{i}{\hbar}S(r,t)\Big]$$

[5] M.K.Parikh and F.Wilczek, Phys. Rev. Lett. 85, 5042 (2000)
[6] R.Banerjee and B.R.Majhi, JHEP 0806, 095 (2008)

Hawking Radiation as A Tunneling Effect

Expanding S(r, t) in a powers of h bar:

$$S(r,t) = S_0(r,t) + \hbar S_1(r,t) + \hbar^2 S_2(r,t) + \ldots = S_0(r,t) + \sum_i \hbar^i S_i(r,t)$$

$$\begin{cases}
\hbar^0 : & \left(\frac{\partial S_0}{\partial t}\right)^2 - f(r)g(r)\left(\frac{\partial S_0}{\partial r}\right)^2 = 0, \\
\hbar^1 : & 2i\frac{\partial S_0}{\partial t}\frac{\partial S_1}{\partial t} - 2if(r)g(r)\frac{\partial S_0}{\partial r}\frac{\partial S_1}{\partial r} - \frac{\partial^2 S_0}{\partial t^2} + f(r)g(r)\frac{\partial^2 S_0}{\partial r^2} \\
& +\frac{1}{2}\Big(f'(r)g(r) + f(r)g'(r)\Big)\frac{\partial S_0}{\partial r} = 0, \\
\hbar^2 : & i\left(\frac{\partial S_1}{\partial t}\right)^2 + 2i\frac{\partial S_0}{\partial t}\frac{\partial S_2}{\partial t} - if(r)g(r)\left(\frac{\partial S_1}{\partial r}\right)^2 - 2if(r)g(r)\frac{\partial S_0}{\partial r}\frac{\partial S_2}{\partial r} - \frac{\partial^2 S_1}{\partial t^2} + f(r)g(r)\frac{\partial^2 S_1}{\partial r^2} \\
& +\frac{1}{2}\Big(f'(r)g(r) + f(r)g'(r)\Big)\frac{\partial S_1}{\partial r} = 0,
\end{cases}$$

It turned out that the solutions of S_i's are proportional to S_0.

$$S(r,t) = S_0(r,t) + \sum_i \beta_i \frac{\hbar^i}{M^{2i}} S_0(r,t) = \Big(1 + \sum_i \beta_i \frac{\hbar^i}{M^{2i}}\Big)\Big[\omega t \pm \omega \int_0^r \frac{dr}{\sqrt{f(r)g(r)}}\Big]$$

Hawking Radiation as A Tunneling Effect

$$\begin{cases} \phi_{\text{in}} = \exp\Big[-\frac{i}{\hbar}(1+\sum_i \beta_i \frac{\hbar^i}{M^{2i}})\Big(\omega t + \omega \int_0^r \frac{dr}{\sqrt{f(r)g(r)}}\Big)\Big] \\ \phi_{\text{out}} = \exp\Big[-\frac{i}{\hbar}(1+\sum_i \beta_i \frac{\hbar^i}{M^{2i}})\Big(\omega t - \omega \int_0^r \frac{dr}{\sqrt{f(r)g(r)}}\Big)\Big] \end{cases}$$

$$\begin{cases} P_{\text{in}} = |\phi_{\text{in}}|^2 \\ P_{\text{out}} = |\phi_{\text{out}}|^2 \end{cases}$$

$$P_{\text{out}} = \exp\Big(-\frac{\omega}{T_h}\Big)P_{\text{in}}$$

$$T_h \;=\; \frac{\hbar}{4}\Big[(1+\sum_i \beta_i \frac{\hbar^i}{M^{2i}})\text{Im}\int_0^r \frac{dr}{\sqrt{f(r)g(r)}}\Big]^{-1} = \frac{\hbar}{8\pi M}\Big(1+\sum_i \beta_i \frac{\hbar}{M^{2i}}\Big)^{-1}$$

Hawking Radiation as A Tunneling Effect – take GUP effect into consideration

Now we start from **the GUP-modified Klein-Gordon equation** :

$$\Box\Phi + 2\beta\hbar^2\Box\Box\Phi + (\frac{mc}{\hbar})^2\Phi = 0$$

Together with the ansatz,

$$\phi(r,t) = \exp\Big[-\frac{i}{\hbar}S(r,t)\Big]$$

Expanding S(r, t) in a powers of $\hbar$:

$$S(r,t) \;=\; S_0(r,t) + \hbar S_1(r,t) + \;\ldots$$

Under Schwarzschild metric the Klein-Gordon eqn. can reduces to :

$$\hbar^0 \;:\; [f^{-1}(r)\Big(\frac{\partial S_0}{\partial t}\Big)^2 - f(r)\Big(\frac{\partial S_0}{\partial r}\Big)^2] + 2\beta[f^{-1}(r)\Big(\frac{\partial S_0}{\partial t}\Big)^2 - f(r)\Big(\frac{\partial S_0}{\partial r}\Big)^2]^2 = 0,$$

$$\hbar^1 \;:\; [2f^{-1}\frac{\partial S_0}{\partial t}\frac{\partial S_1}{\partial t} - 2f\frac{\partial S_0}{\partial r}\frac{\partial S_1}{\partial r}] + i[f^{-1}\frac{\partial^2 S_0}{\partial t^2} + f(r)\frac{\partial^2 S_0}{\partial r^2} - f'(r)\frac{\partial S_0}{\partial r}] + 2\beta\{a\,lot\,of\,terms\,depend\,on\,S_0S_1\}$$

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Hawking Radiation as A Tunneling Effect – take GUP effect into consideration

(1): $[f^{-1}(r)\left(\frac{\partial S_0}{\partial t}\right)^2 - f(r)\left(\frac{\partial S_0}{\partial r}\right)^2] = 0$ ➡ $\widetilde{S}_1 = \widetilde{S}_0 = \left[\omega t \pm \omega \int_0^r \frac{dr}{f(r)}\right]$

(2): $1 + 2\beta[f^{-1}(r)\left(\frac{\partial S_0}{\partial t}\right)^2 - f(r)\left(\frac{\partial S_0}{\partial r}\right)^2] = 0$

➡
$$\begin{cases} \widetilde{S}_0^{New} = \left[\omega t \pm \omega \int_0^r dr \frac{1}{f(r)}\left(\sqrt{1 + \frac{f}{2\beta\omega^2}}\right)\right] \\ \widetilde{S}_1^{New} = \left[\omega t \pm \omega \int_0^r dr \frac{\omega/f + \beta[\pm 4K^3/f^2 - 4\omega^3 J^2] \mp i(fJ)'/2 \pm i\beta(6\omega^2 f^2 J^2 J' + 4\omega^2 f f' J^3 - 2\omega^2 J' - 8JKK')}{J[f + 4\beta\omega^2(1 - f^2 J^2)]}\right] \end{cases}$$

where $J(r) \equiv \frac{1}{f}\sqrt{1 + \frac{f}{2\beta\omega^2}}$ $K(r) \equiv \sqrt{\omega^2 f^2 J^2 + \frac{1}{2\beta}}$

$$S(t,r) = \beta_0 \widetilde{S}_0 + \beta_1 \hbar \widetilde{S}_1 + \alpha_0 \widetilde{S}_0^{New} + \alpha_1 \hbar \widetilde{S}_1^{New} + O(\hbar^2) + O(\beta^2)$$

Hawking Radiation as A Tunneling Effect – correction to black hole temperature

$$\left\{\begin{aligned}\frac{P_{\rm out}}{P_{\rm in}} &= \exp\Big(-8\pi M\omega[(\alpha_0+\beta_0)+\hbar(\frac{\alpha_1}{3}+\beta_1)] - \hbar\frac{64}{3}\alpha_1\beta M\pi\omega^3\Big)\\ &\sim \exp\Big(-8\pi M\omega[(\alpha_0+\beta_0)+\hbar(\frac{\alpha_1}{3}+\beta_1)] - \hbar\frac{64}{3}\alpha_1\beta M\pi <\omega^2> \omega\Big)\\ \frac{P_{\rm out}}{P_{\rm in}} &= \exp\Big(-\frac{\omega}{T_h}\Big)\end{aligned}\right.$$

$$\Rightarrow\quad T_h = [(\alpha_0+\beta_0)+\hbar(\frac{\alpha_1}{3}+\beta_1)]^{-1}\frac{1}{8\pi M}\Big(1-\beta\frac{\alpha_1\hbar/3}{[(\alpha_0+\beta_0)+\hbar(\frac{\alpha_1}{3}+\beta_1)]}\Big)^{-1}$$

$$\approx \boxed{T_0 - (\frac{\alpha_1}{3}+\beta_1)T_0\hbar - \frac{16}{3}\alpha_1 T_0^3\beta\hbar} + O(\hbar^2) + O(\beta^2)$$

where $T_0 = \dfrac{1}{8\pi M}$

Summary

- By invoking the Hamilton-Jacobi method beyond the semiclassical approximation together with the GUP-inspired Klein-Gordon equation, we compute the corrected Hawking temperature.
- Finally we find that the 1st order quantum correction is no longer proportion to the semiclassical contribution as usual and thus the black hole radiation is no longer the perfect black body spectrum.
- The approximated black hole temperature with a GUP correction proportional to β is derived which is consistent with previous works in the literature.

Thanks for your attention

temp

$$X^i = (1 + \beta \boldsymbol{p}^2) x^i + \beta' p^i (\boldsymbol{p} \cdot \boldsymbol{x}) + \mathrm{i}\hbar\gamma p^i, \qquad P^i = p^i, \quad \text{where } x^i = \mathrm{i}\hbar\partial/\partial p^i$$

$$T_h = \left(\frac{1}{8\pi M}\right)\left[(\alpha_0 + \beta_0) + \hbar(\frac{\alpha_1}{3} + \beta_1)\right]^{-1} \times \left(1 - \frac{\beta\alpha_1\hbar}{3[(\alpha_0+\beta_0) + \hbar(\alpha_1/3 + \beta_1)]}\right)^{-1}$$

$$\approx T_0(\alpha_0+\beta_0)^{-1}\Big[1 - (\alpha_0+\beta_0)^{-1}(\frac{\alpha_1}{3} + \beta_1)\hbar$$

$$-(\alpha_0+\beta_0)^{-1}(\frac{16}{3}\alpha_1\beta T_0^2)\hbar + O(\hbar^2) + O(\beta^2)\Big]$$

$$\approx T_0 - (\frac{\alpha_1}{3} + \beta_1)T_0\hbar - \frac{16}{3}\alpha_1 T_0^3 \beta\hbar + O(\hbar^2) + O(\beta^2)$$

The Ghost Problem

In principle the GUP-inspired Klein-Gordon equation which is a fourth-order field equation would induce ghost modes in its solutions, which is generally perceived as problematic.

Our GUP correction to the BH temperature is actually induced by the ghost solution of the GUP-inspired Klein-Gordon equation.

However, since the only scale associated with GUP is the Planck scale, it is reasonable to assign the pole associated with the ghost to the Planck scale. Therefore in the weak gravity limit, the ghost problem should be harmless.