Holographic Entanglement Entropy in Insulator/Superconductor Transition

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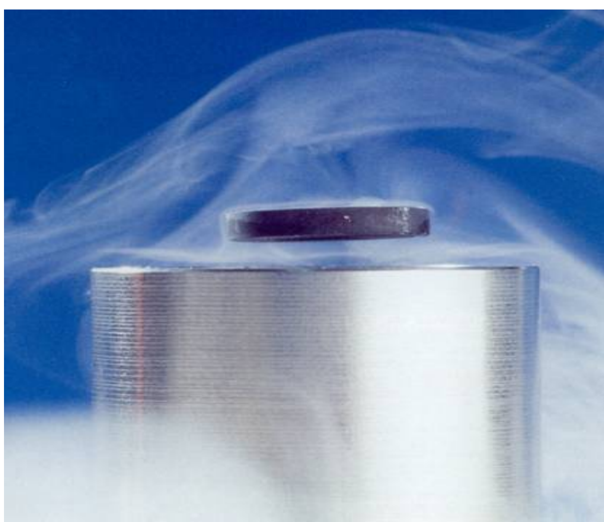
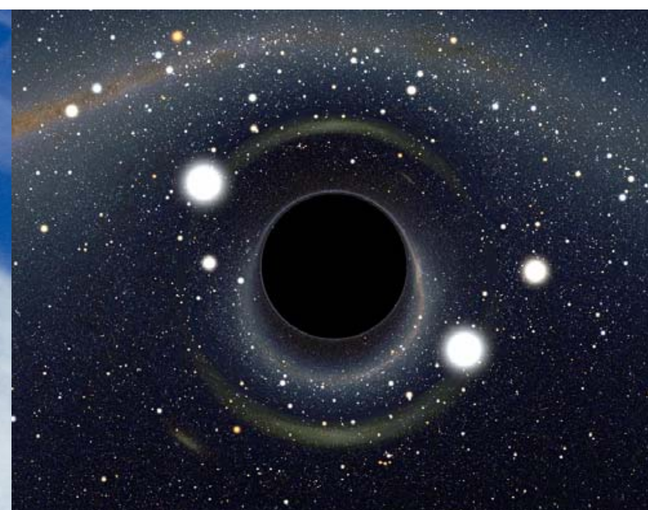
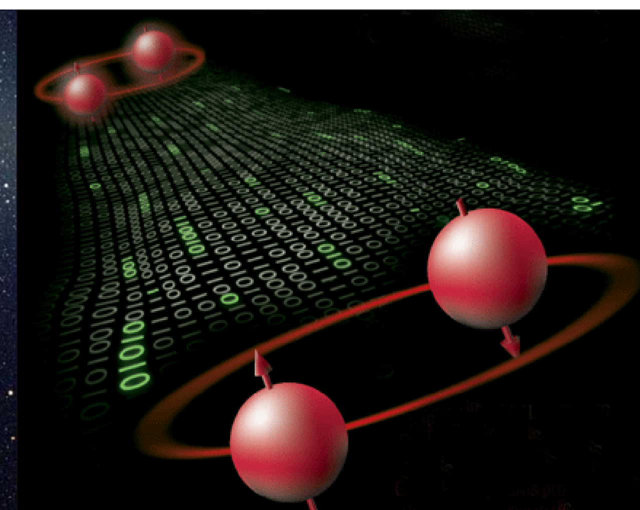
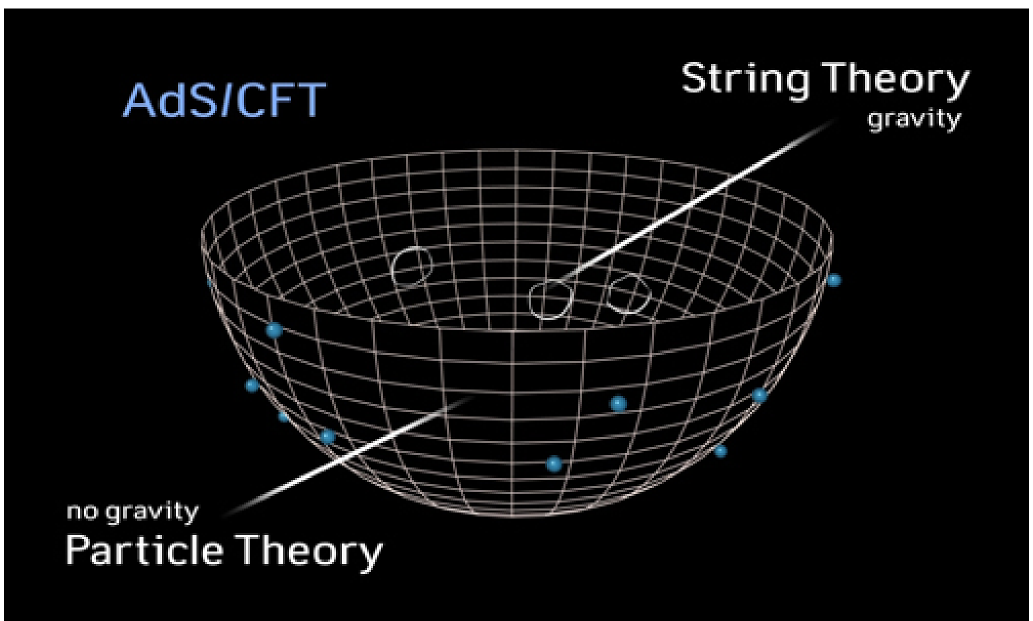
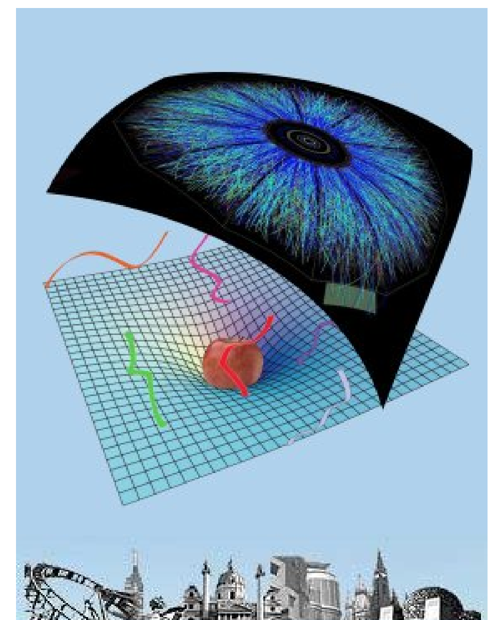




Outline

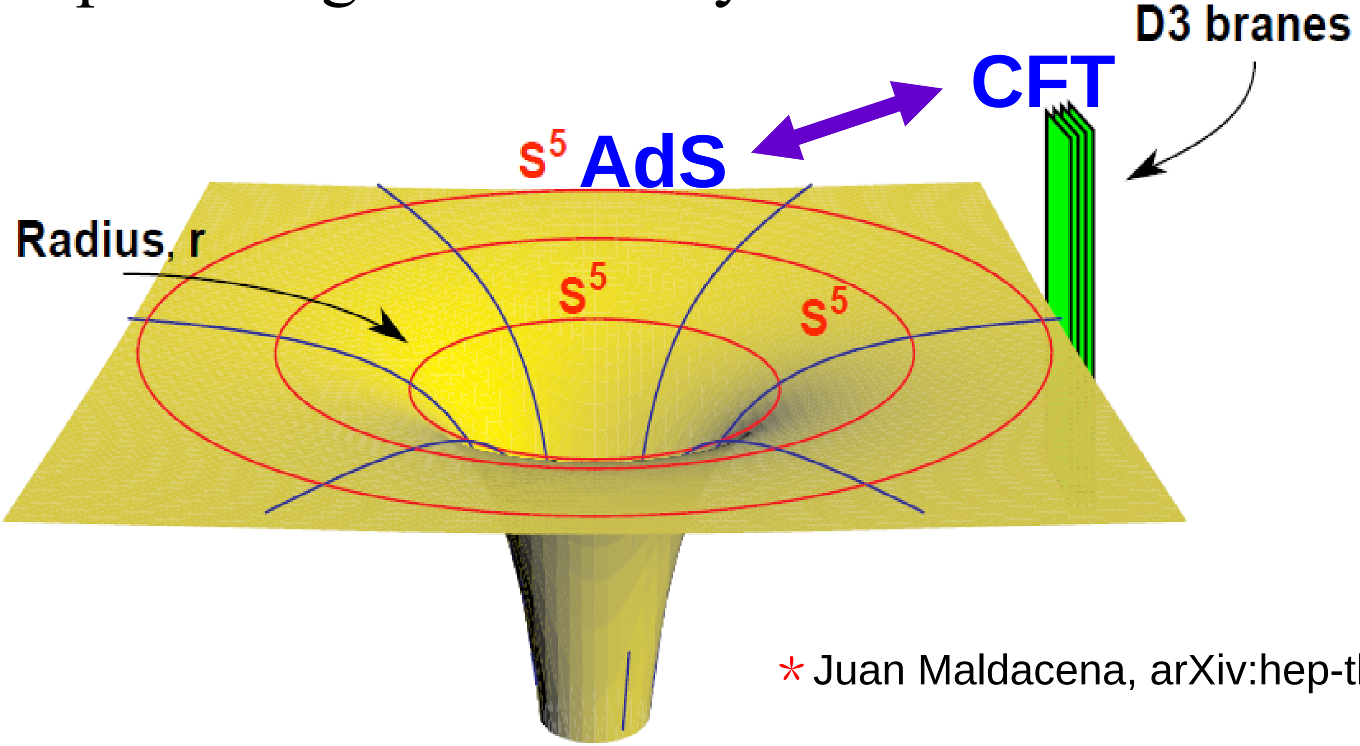
- Basic Concepts in AdS/CFT Correspondence
- Insulator/Superconductor Phase Transition
- Entanglement Entropy and Holographic Formula
- Conclusions





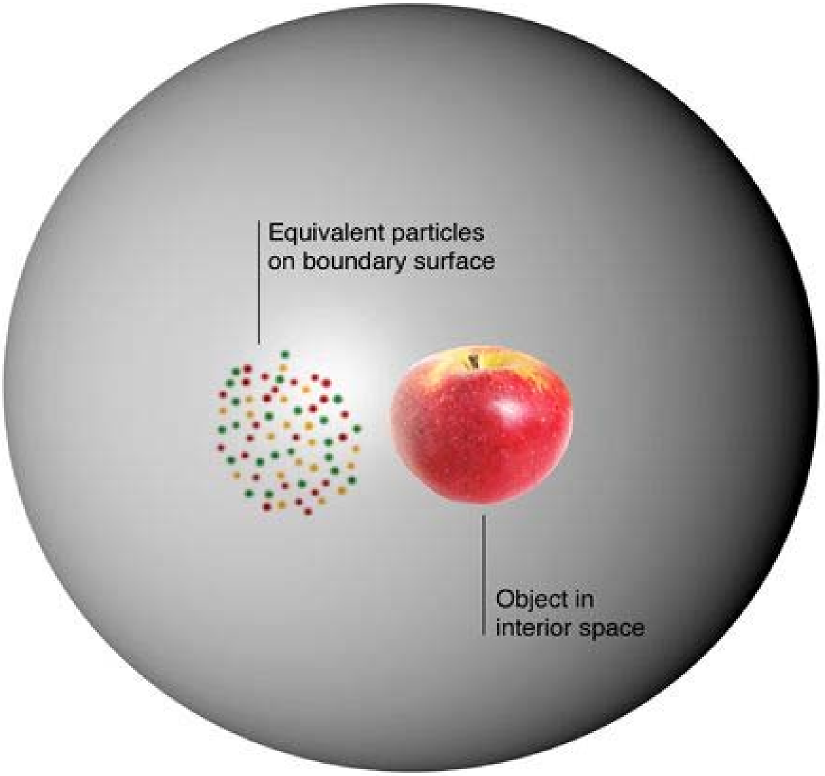
AdS/CFT Duality

- In 1997 Juan Maldacena first conjectured that typeIIB string theory on AdS5 $\times$S5 should be somehow dual to the $\mathcal{N}=4$ U(N) super-Yang-Mills theory in 3+1 dimensions.



*Juan Maldacena, arXiv:hep-th/9711200

The concrete realization of holographic principle



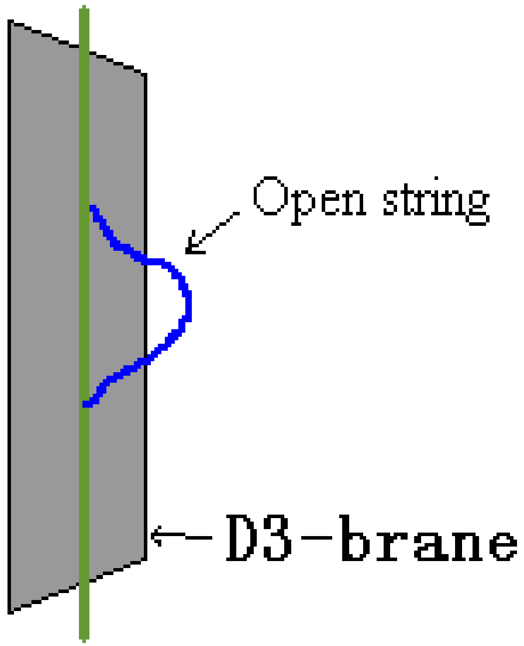
→ Quantum Gravity !?

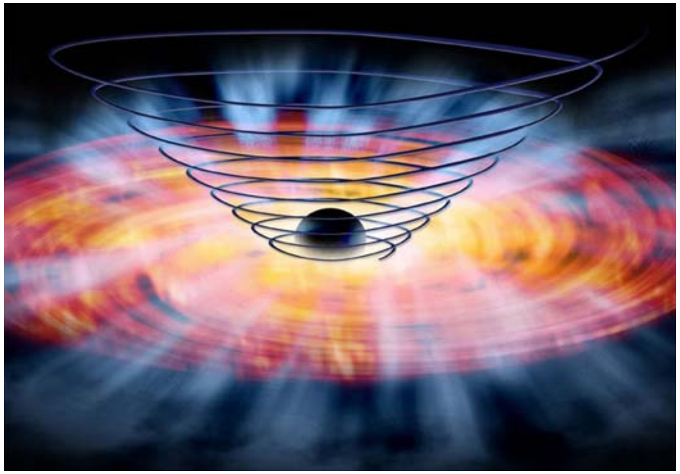
*G. t Hooft, arXiv:hep-th/9310026

*L. Susskind, arXiv:hep-th/9409089

AdS/CFT is a strong/weak coupling duality

Weak coupling ⟷ Strong coupling





The formula proposed by Witten, Polyakov, etc in 1998:

$$\langle e^{\int d^4x \phi_0(\vec{x})\mathcal{O}(\vec{x})}\rangle_{CFT} = \mathcal{Z}_{string}\left[\phi(\vec{x},z)\Big|_{z=0} = \phi_0(\vec{x})\right]$$

This relationship is usually taken as the backbone of the AdS/CFT duality.

Operator in the dual field theory ⟷ **Dynamical field propagating on bulk**

AdS/CFT dictionary

* E. Witten, arXiv:hep-th/9802150

* S. S. Gubser, I. R. Klebanov, A. M. Polyakov, arXiv:hep-th/9802109

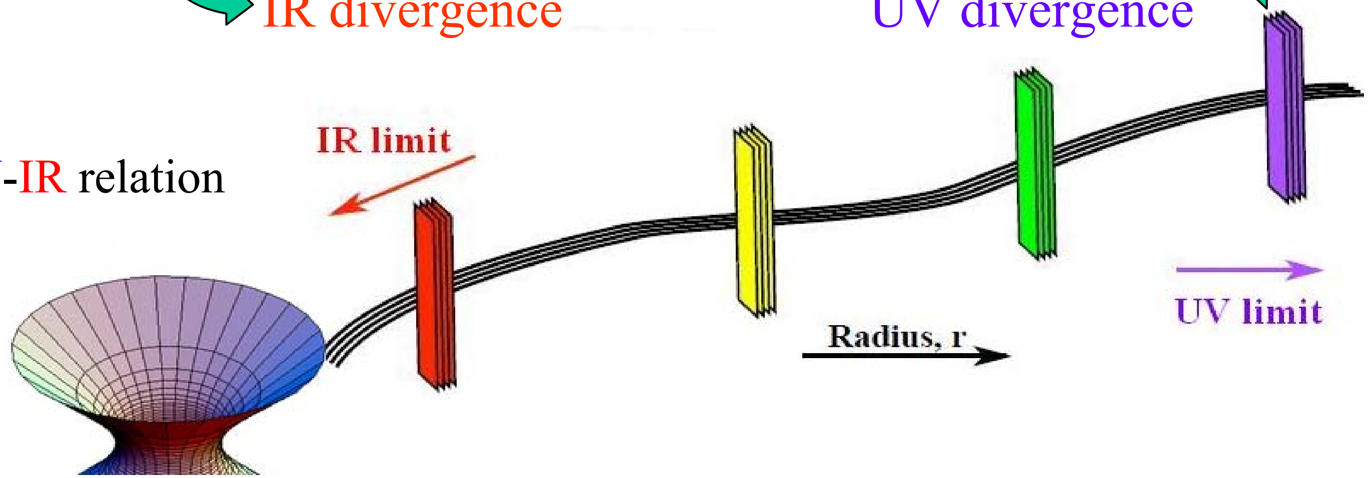
Holographic Renormalization

In the leading saddle point approximation:

$$S_{onshell}[\phi_0(x)] = W_{QFT}[\phi_0(x)]$$

IR divergence UV divergence

UV-IR relation

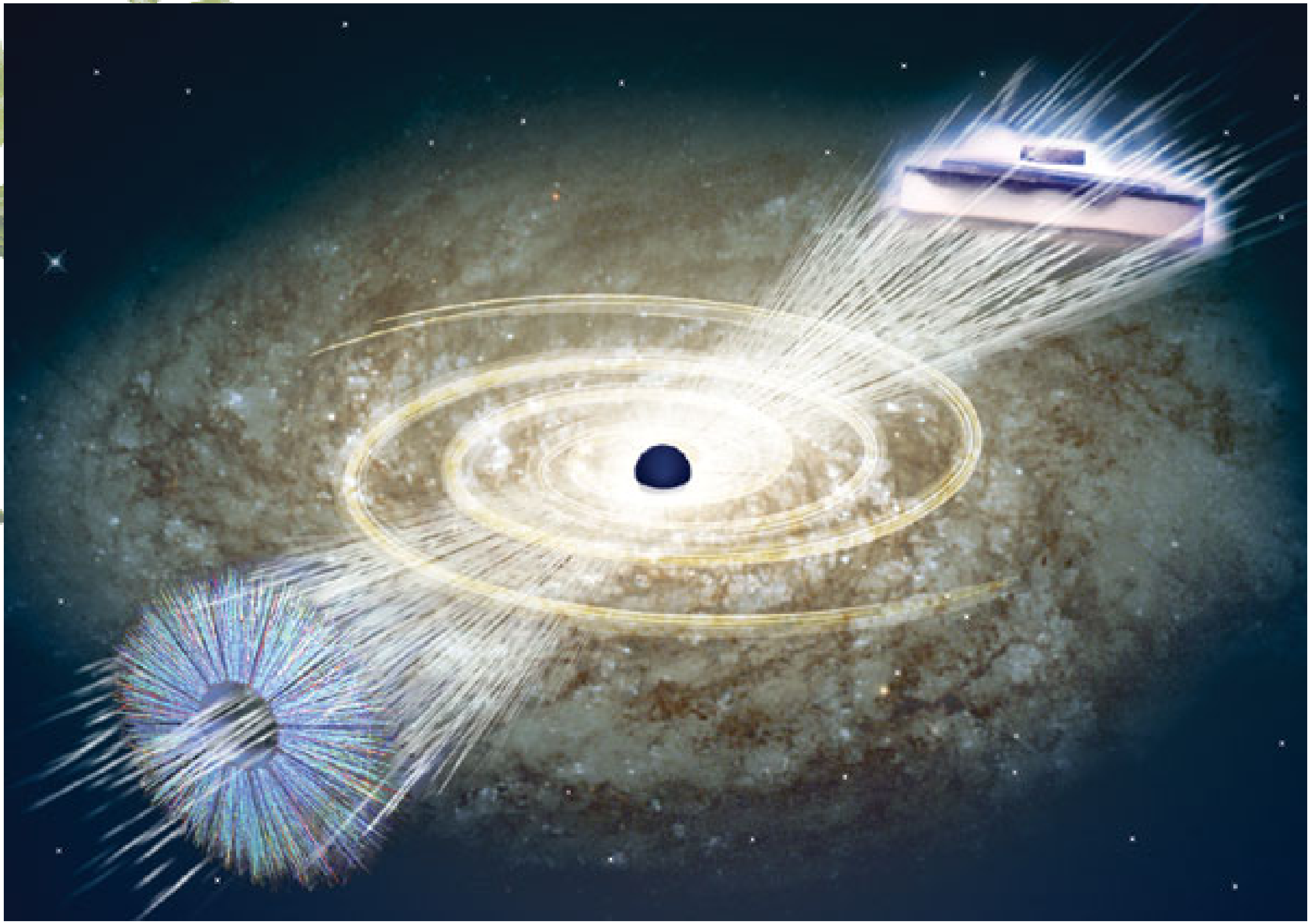


Introduce covariant counterterms in gravity side

$$\langle O(x) \rangle_s = \frac{1}{\sqrt{g_{(0)}(x)}} \frac{\delta S_{ren}}{\delta \phi_{(0)}(x)}$$

RG transformation can be studied by using bulk diffeomorphisms that induces a Weyl transformations on the boundary metric.

* S. Susskind, E. Witten, arXiv:hep-th/9805114 * K. Skenderis, arXiv:hep-th/0209067



the duality has been generalized to gravitational theories with certain other boundary conditions, and to field theories that are not conformally invariant.

* G. T. Horowitz, J. Polchinsk, arXiv:gr-qc/0602037

Two Complementary Approaches

Bottom to up

※ Toy models coming from simple gravity theory;

※ Basic ingredients: $g_{\mu\nu}, A_\mu, \Psi, \Phi \ldots$

※ Advantage: simplicity and universality;

※ Disadvantage: the dual field theory is unclear.

Top to down

※ Configurations originated from string/M theory;

※ Exact solutions of supergravity or Dp/Dq-branes;

※ Advantage: good understanding on field theory;

※ Disadvantage: complexity.

Two Main Methods

Retarded Green's function method

- ★ General, leading to many transports coefficients;
- ★ Retarded Green's function in bulk encodes a retarded correlator of its dual field operator;
- ★ Kubo's formula ⟶ transports coefficients .

* D.T.Son, A.O.Starinets, arXiv:0205051

The membrane paradigm

- ★ Hydrodynamic behavior of boundary field theory VS those at stretched horizon of the black hole;
- ★ Transport coefficients VS quantities at stretched horizon ;
- ★ Elegantly explains universalities of Transport coefficients .

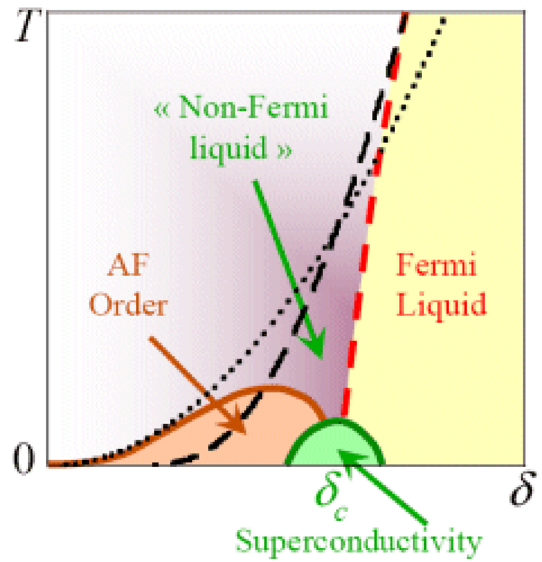
* N. Iqbal and Hong Liu, arXiv:0809.3808

AdS/CFT and Condensed Matter

The holographic duality is a powerful theoretical method to investigate strongly coupled field theories. In condensed matter physics, there are many strongly interacting systems that can be engineered and studied in detail in laboratories:

- **quantum phase transition**
- **superfluidity and superconductivity**
- **cold atoms**
- **fermi liquid and non-fermi liquid**

……



This holographic methods may be able to offer insight into some of these nonconventional materials, thus in turn leading to experimental AdS/CFT.

* C. P. Herzog, arXiv:hep-th/0904.1975

* S. A. Hartnoll, arXiv:hep-th/0903.3246

Compare to real experimental data in graphene

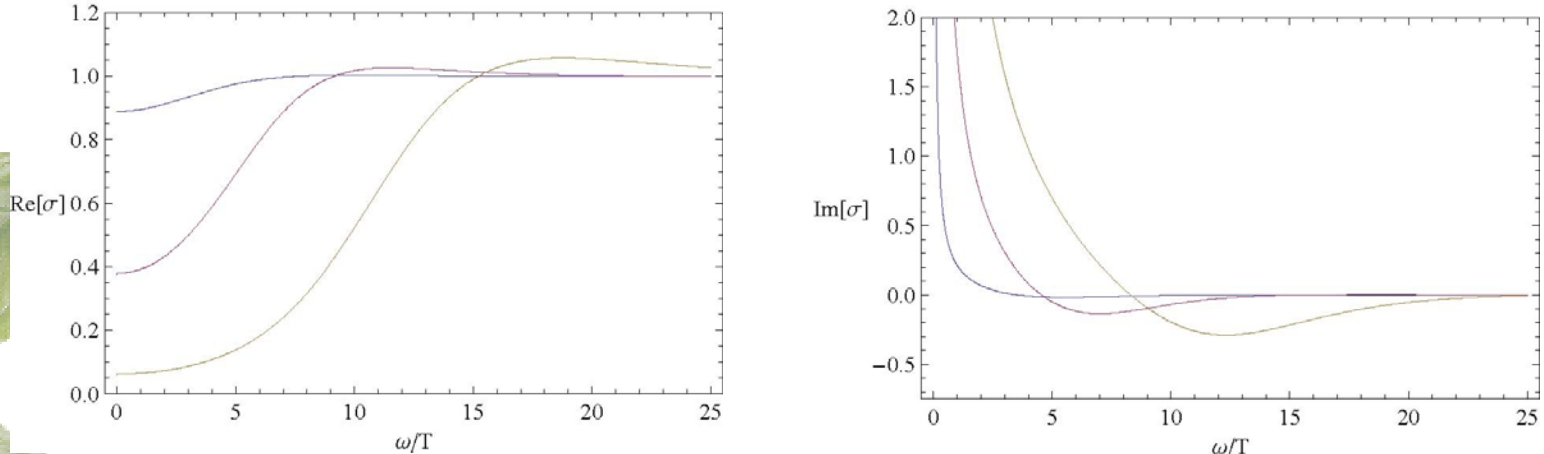


Figure: Left: Plots of the real part of the electrical conductivity vs. frequency via AdS/CFT.
Right: Plots of the imaginary part of the electrical conductivity vs. frequency.
Different curves dual to different values of the chemical potential at fixed temperature.

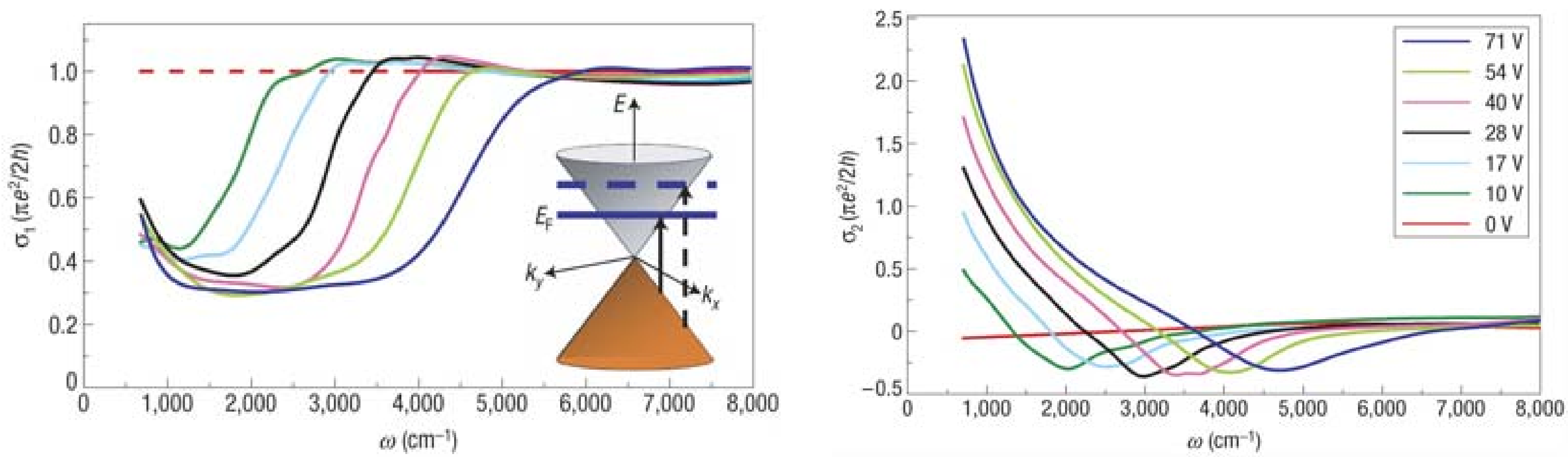


Figure: Experimental plots of the real (Left) and imaginary (Right) part of the electrical conductivity in graphene as a function of frequency.
Different curves correspond to different values of the gate voltage at fixed temperature.

* Z. Q. Li, E. A. Henriksen, etc, Nature Physics 4, 532 - 535 (2008)

Holographic Superconductivity

Einstein-Maxwell-charged scalar theory with a negative cosmological constant:

$$S = \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-g}(R + \frac{d(d-1)}{L^2} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |\nabla_\mu\psi - iqA_\mu\psi|^2 - m^2|\psi|^2)$$

Scalar serves as an order parameter . The normal phase corresponds to RN-AdS black hole background with vanishing scalar field.

When the temperature of the black hole is below a critical temperature, there are at least two distinct mechanisms leading to the black hole solution unstable to develop a scalar hair which behaves near the boundary as

$$\psi = \frac{\psi^{(1)}}{r^{\Delta_-}} + \frac{\psi^{(2)}}{r^{\Delta_+}} + \dots$$

This results in a superconductor (superfluid) phase transition.

* S. A. Hartnoll, C. P. Herzog, G. T. Horowitz, arXiv:0810.1563

A Physical Picture

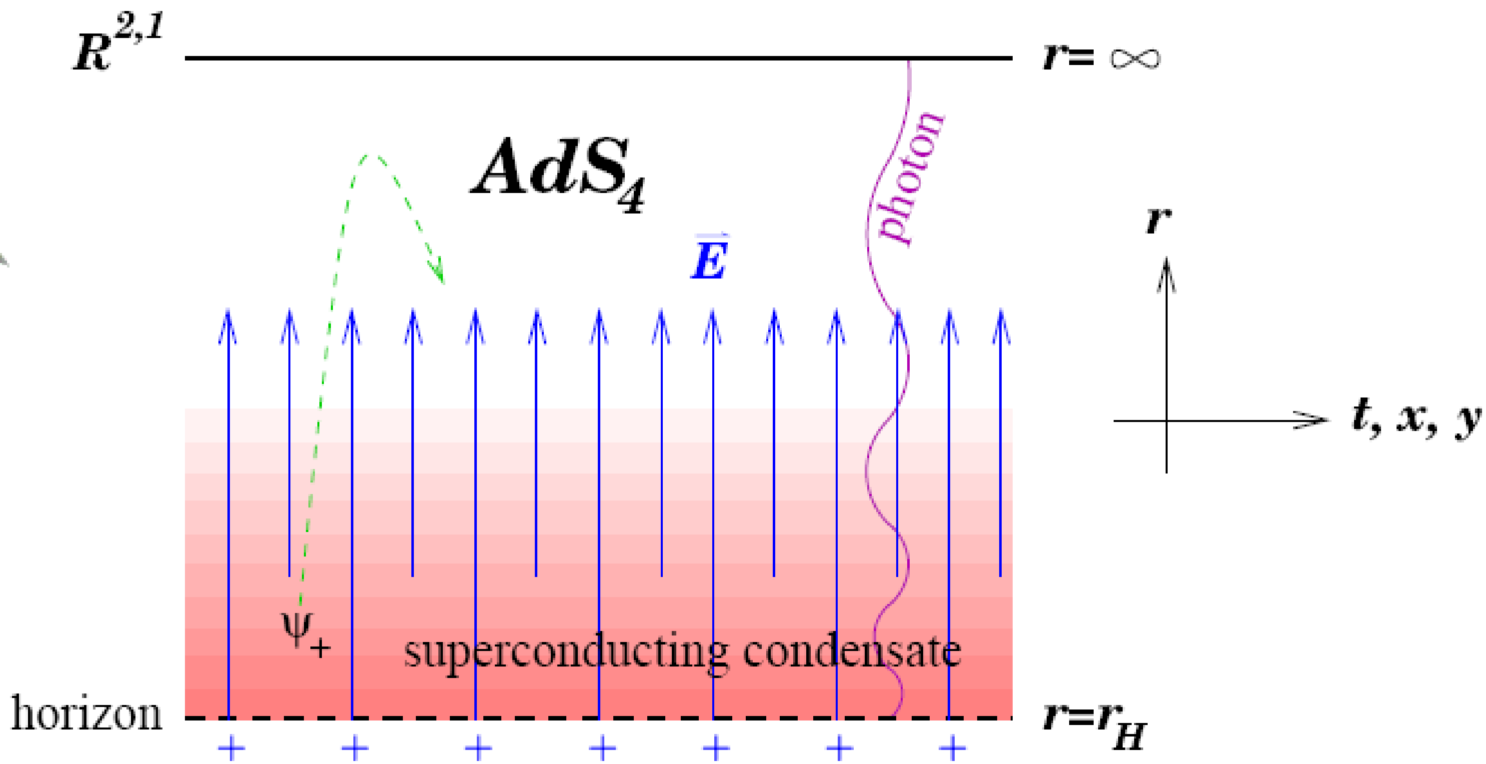


Figure: A superconducting condensate floats above a black hole horizon because of a balance of gravitational and electrostatic forces.

* Steven S. Gubser, Silviu S. Pufu, arXiv:0805.2960

The Superconducting Phase Transition

Given $d = 3, m^2 = -2$ (above the BF bound), we can choose a scalar in the field theory with scaling dimension one or two.

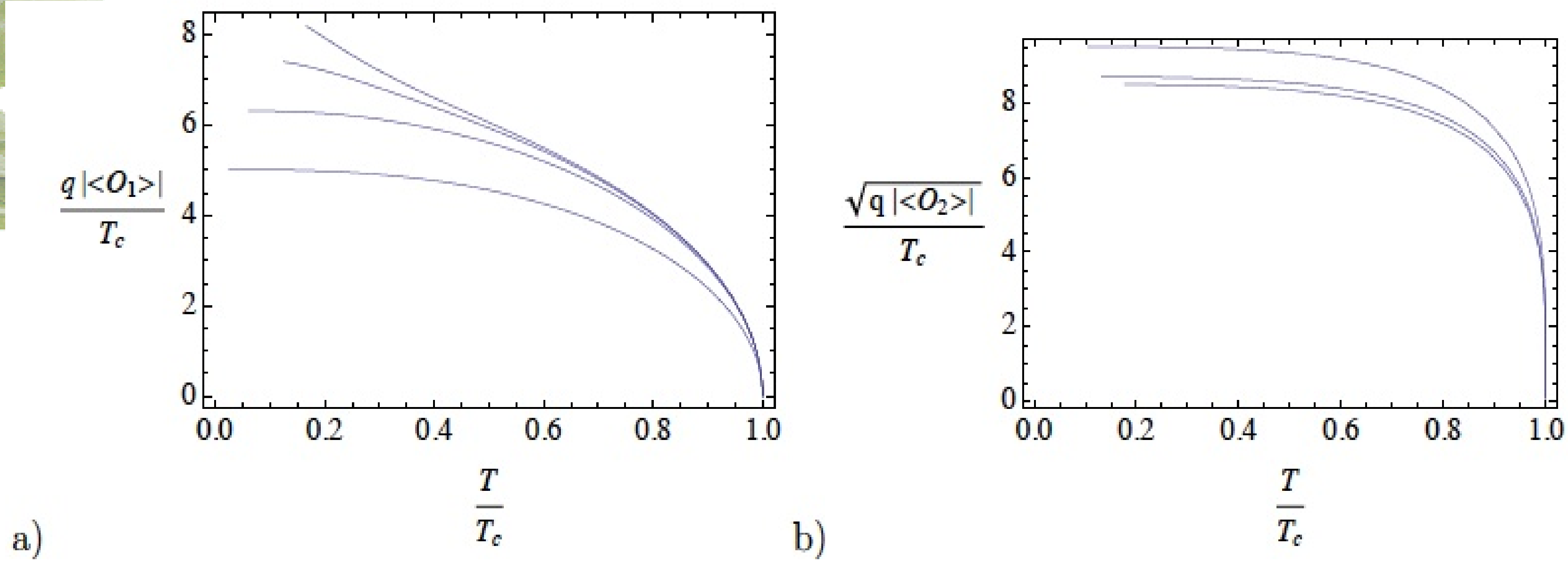


Figure: Plots of temperature vs. condensate for operators with dimension one and two.

For $T \lesssim T_c$,

$$\langle \mathcal{O}_i \rangle \sim (T_c - T)^{1/2} \, ,$$

the standard Landau-Ginzburg mean field result.

Conductivity

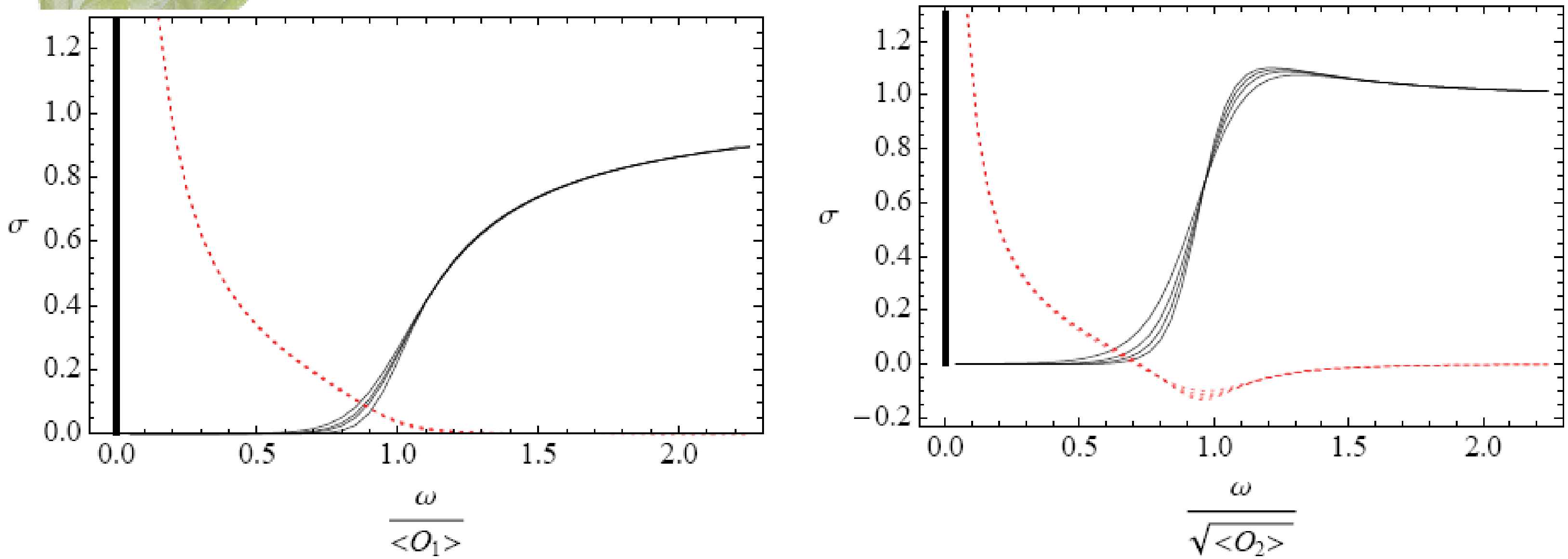


Figure: Plots of the real part of the conductivity vs. frequency at low temperature for dimension one case (left) and dimension two case (right). The dotted red curve is the imaginary of the conductivity.

Re$[\sigma(\omega)]$ contains a delta function $\pi n_s \delta(\omega)$ which leads to superconductivity where n_s is the superfluid density.

Evidence for strong binding: $\langle \mathcal{O}_1 \rangle$ and $\sqrt{\langle \mathcal{O}_2 \rangle}$ can be reinterpreted as twice the superconducting gap Δ.

Recall that: For BCS at $T = 0$, $2\Delta = 3.54\, T_c$.

Insulator/Superconductor Phase Transition

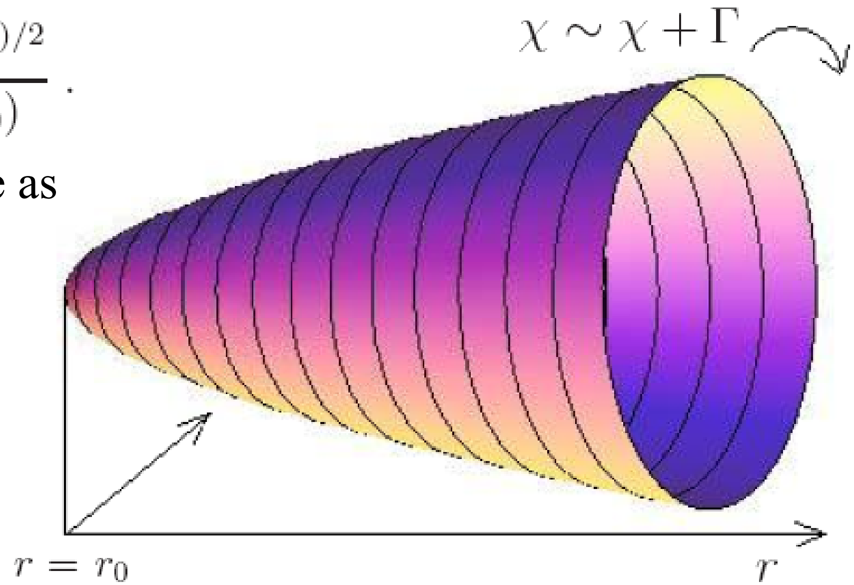
Consider the back reaction of the matter , we choose the ansatz

$$ds^2 = \frac{dr^2}{r^2 B(r)} + r^2 \left(-e^{C(r)} dt^2 + dx^2 + dy^2 + e^{A(r)} B(r) d\chi^2\right)$$

$$A_t = \phi(r), \quad \psi = \psi(r)$$

Here we have set $L = 1$ without lose of generality. We require that $B(r)$ vanishes at the tip of the soliton. And in order to obtain a smooth geometry at the tip r_0, χ should be made with an identification

$$\chi \sim \chi + \Gamma, \qquad \Gamma = \frac{4\pi e^{-A(r_0)/2}}{r_0^2 B'(r_0)} .$$



The matter fields near the boundary r → ∞ behave as

$$\psi = \frac{\psi^{(1)}}{r^{\Delta_-}} + \frac{\psi^{(2)}}{r^{\Delta_+}} + \dots, \qquad \phi = \mu - \frac{\rho}{r^2} + \dots$$

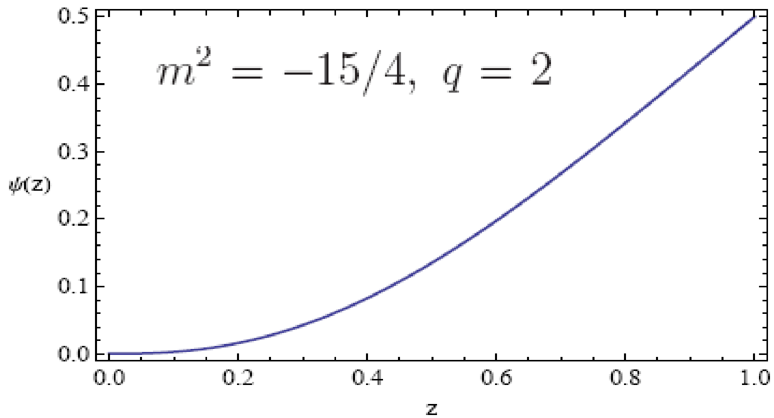
To recover the pure AdS boundary, we also need

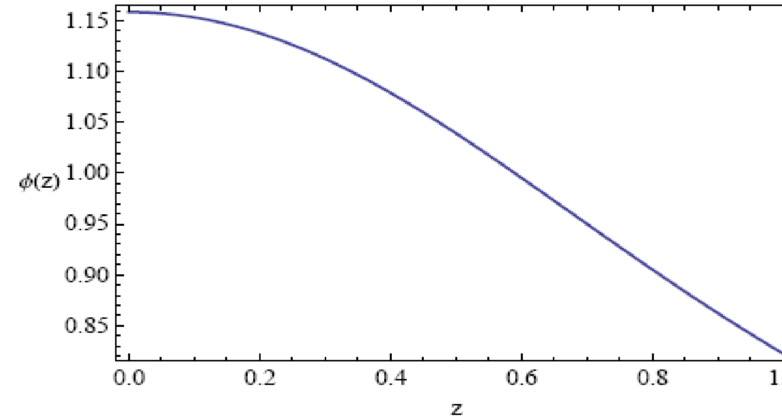
$$A(\infty) = 0 \text{ and } C(\infty) = 0.$$

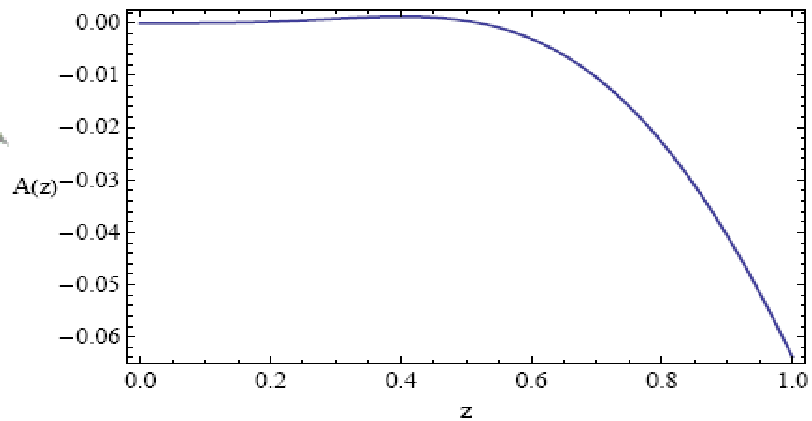
* T. Nishioka, S. Ryu , T. Takayanag, arXiv:0911.0962

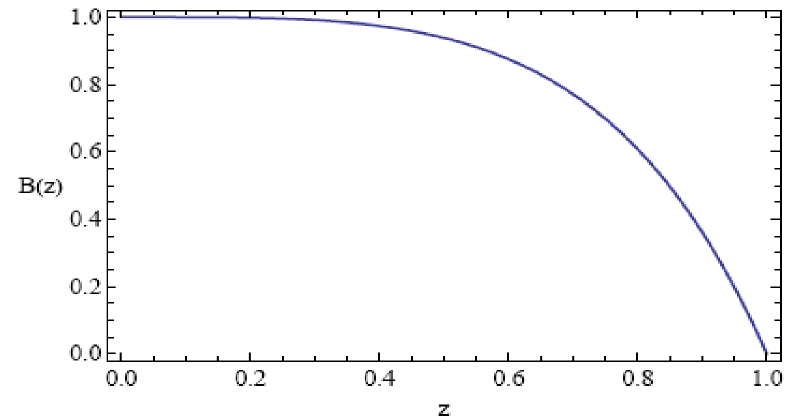
* R.G. Cai, S. He, L. Li, Y.L. Zhang, arXiv:1203.6620

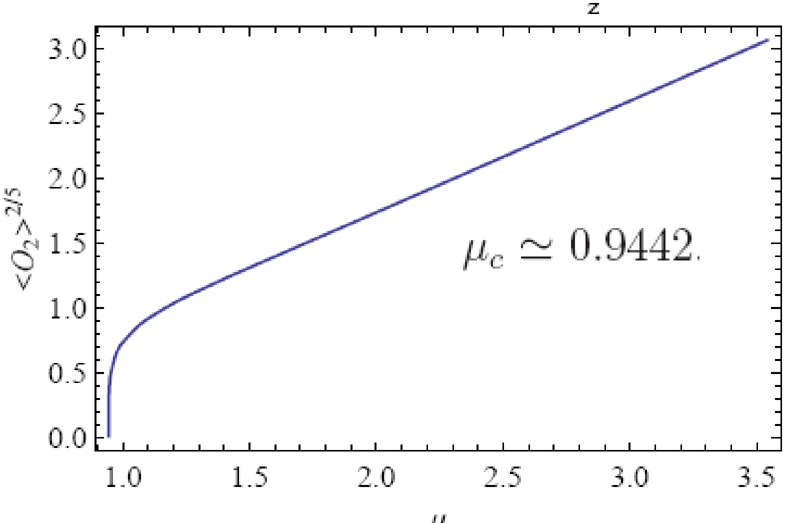
Numerical Results

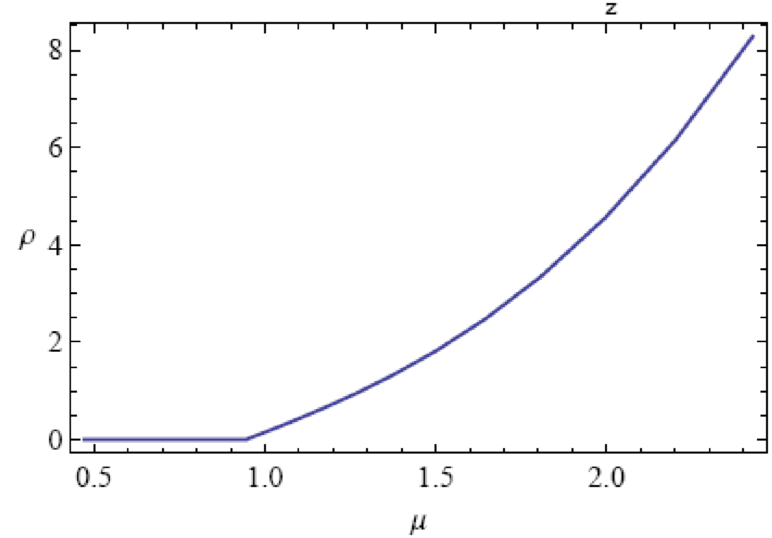






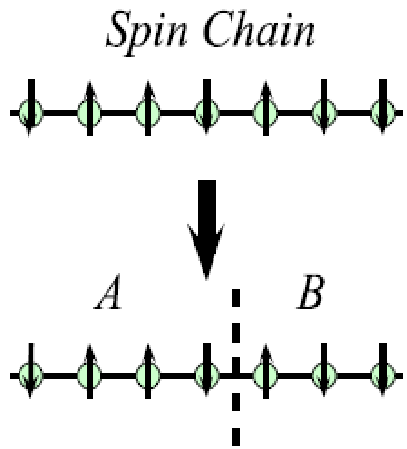


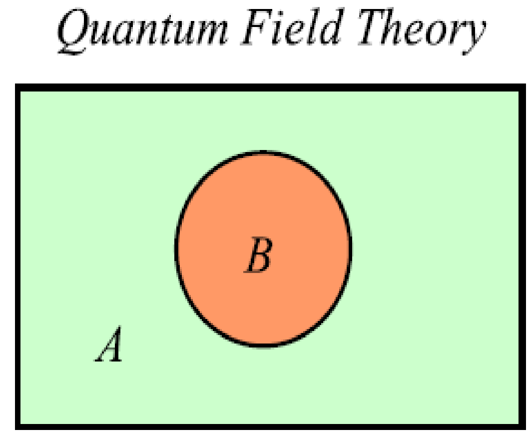




Entanglement Entropy

◆ divide quantum system into two parts A and B



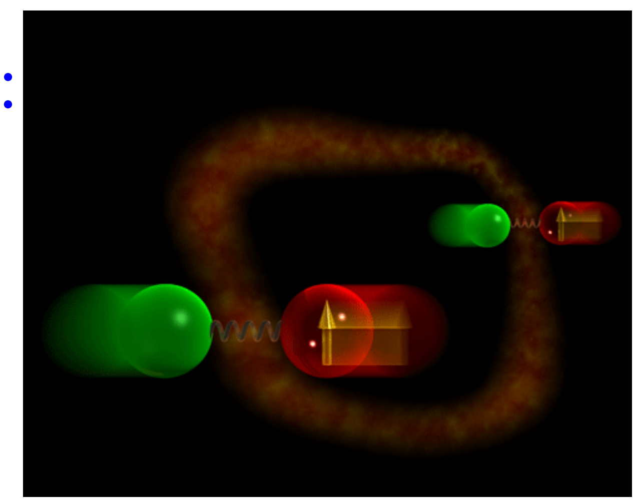


◆ trace over the degrees of freedom in region B

◆ remaining degrees of freedom are described by a density matrix ρ_A

→ calculate von Neumann entropy:

$$S_{EE} = -Tr\left[\rho_A \log \rho_A\right]$$



Area Law

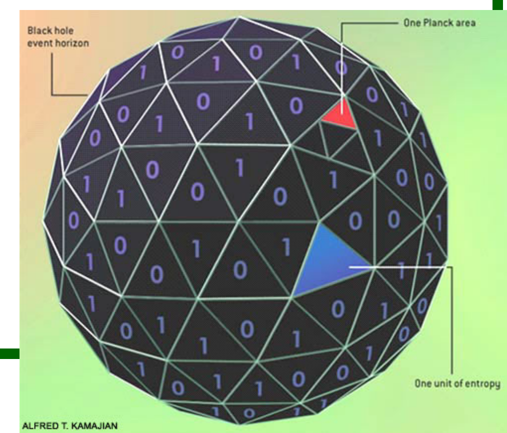
EE in QFTs includes UV divergences.

Area Law

In a (d+1) dim. QFT with a UV fixed point, the leading term of EE is proportional to the area of the (d-1) dim. boundary ∂A :

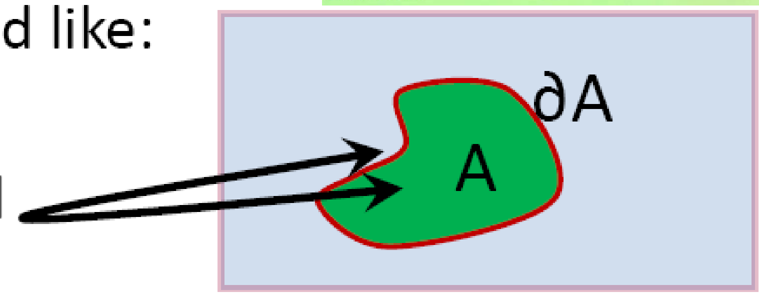
$$S_A \sim \frac{\text{Area}(\partial A)}{a^{d-1}} + (\text{subleading terms}),$$

where a is a UV cutoff (i.e. lattice spacing).



Intuitively, this property is understood like:

Most strongly entangled



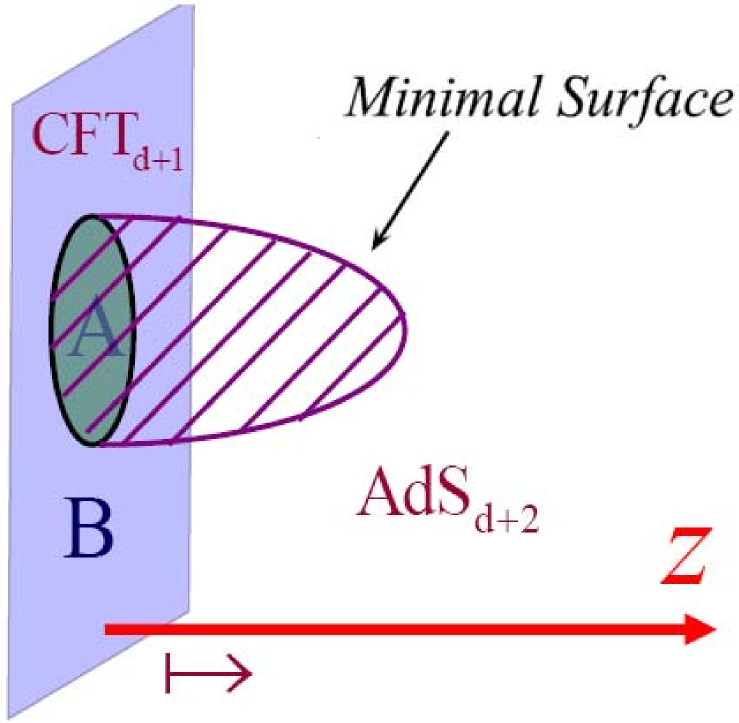
Holographic Entanglement Entropy

Holographic Entanglement Entropy Formula :

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$

γ_A is the minimal area surface (codim.=2) such that

$$\partial A = \partial \gamma_A \ \text{and} \ A \sim \gamma_A \ .$$

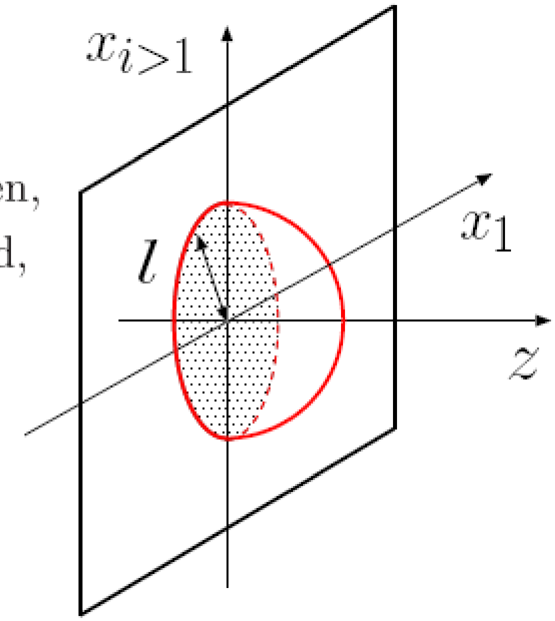


$z > a$ (UV cut off)

* S. Ryu, T. Takayanagi, arXiv:hep-th/0603001

Entanglement Entropy for Circular Disk

$$S_{A_D} = \frac{2\pi^{d/2}R^d}{4G_N^{(d+2)}\Gamma(d/2)}\int_{a/l}^{1} dy \frac{(1-y^2)^{(d-2)/2}}{y^d}$$
$$= p_1\,(l/a)^{d-1} + p_3\,(l/a)^{d-3} + \cdots$$
$$\cdots + \begin{cases} p_{d-1}\,(l/a) + p_d + \mathcal{O}(a/l), & d\text{: even}, \\ p_{d-2}\,(l/a)^2 + q\log(l/a) + \mathcal{O}(1), & d\text{: odd}, \end{cases}$$



where the coefficients are defined by

$$p_1/C = (d-1)^{-1}, \quad p_3/C = -(d-2)/[2(d-3)], \quad \cdots$$
$$p_d/C = (2\sqrt{\pi})^{-1}\Gamma(d/2)\Gamma\left((1-d)/2\right) \quad (\text{if } \ d = \text{even}),$$
$$q/C = (-)^{(d-1)/2}(d-2)!!/(d-1)!! \quad (\text{if } \ d = \text{odd}),$$
$$\text{where} \quad C \equiv \frac{\pi^{d/2}R^d}{2G_N^{d+2}\Gamma(d/2)}.$$

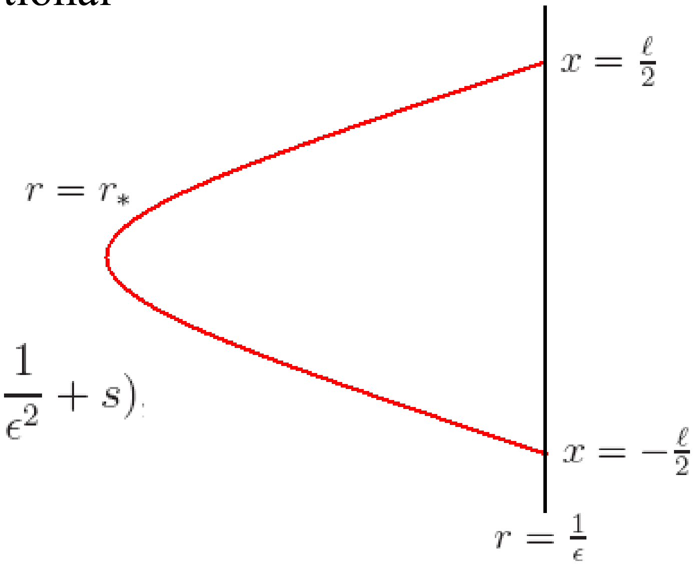
$$ds^2 = R^2\,\frac{dz^2 - dx_0^2 + \sum_{i=1}^{d} dx_i^2}{z^2}$$

* Shinsei Ryu, Tadashi Takayanagi, arXiv:hep-th/0605073

HEE for Belt Geometry in Our Holographic Model

For smooth case, we need to minimize the following functional

$$S_{\mathcal{A}}[x] = \frac{R\Gamma}{2G_N}\int_{r_*}^{\frac{1}{\epsilon}} r e^{\frac{A(r)}{2}}\sqrt{1+r^4B(r)(dx/dr)^2}dr$$



we obtain the entanglement entropy as

$$S_{\mathcal{A}} = \frac{R\Gamma}{2G_N}\int_{r_*}^{\frac{1}{\epsilon}} \frac{r^4\sqrt{B(r)}e^{A(r)}}{\sqrt{r^6B(r)e^{A(r)}-r_*^6B(r_*)e^{A(r_*)}}}dr = \frac{R\Gamma}{4G_N}(\frac{1}{\epsilon^2}+s)$$

The belt width is given by

$$\frac{\ell}{2} = \int_{r_*}^{\frac{1}{\epsilon}}\frac{dx}{dr}dr = \int_{r_*}^{\frac{1}{\epsilon}}\frac{1}{r^2\sqrt{B(r)(\frac{r^6B(r)e^{A(r)}}{r_*^6B(r_*)e^{A(r_*)}}-1)}}dr$$

there is also a disconnected solution described as two separated surfaces

$$S_{\mathcal{A}}^{dis} = 2\frac{R\Gamma}{4G_N}\int_{r_0}^{\frac{1}{\epsilon}} r e^{\frac{A(r)}{2}}dr = \frac{R\pi}{4G_N}(\frac{1}{\epsilon^2}+s)$$

* Rong-Gen Cai, Song He, Li Li, Yun-Long Zhang, arXiv:1203.6620

“confinement/deconfinement” phase transition

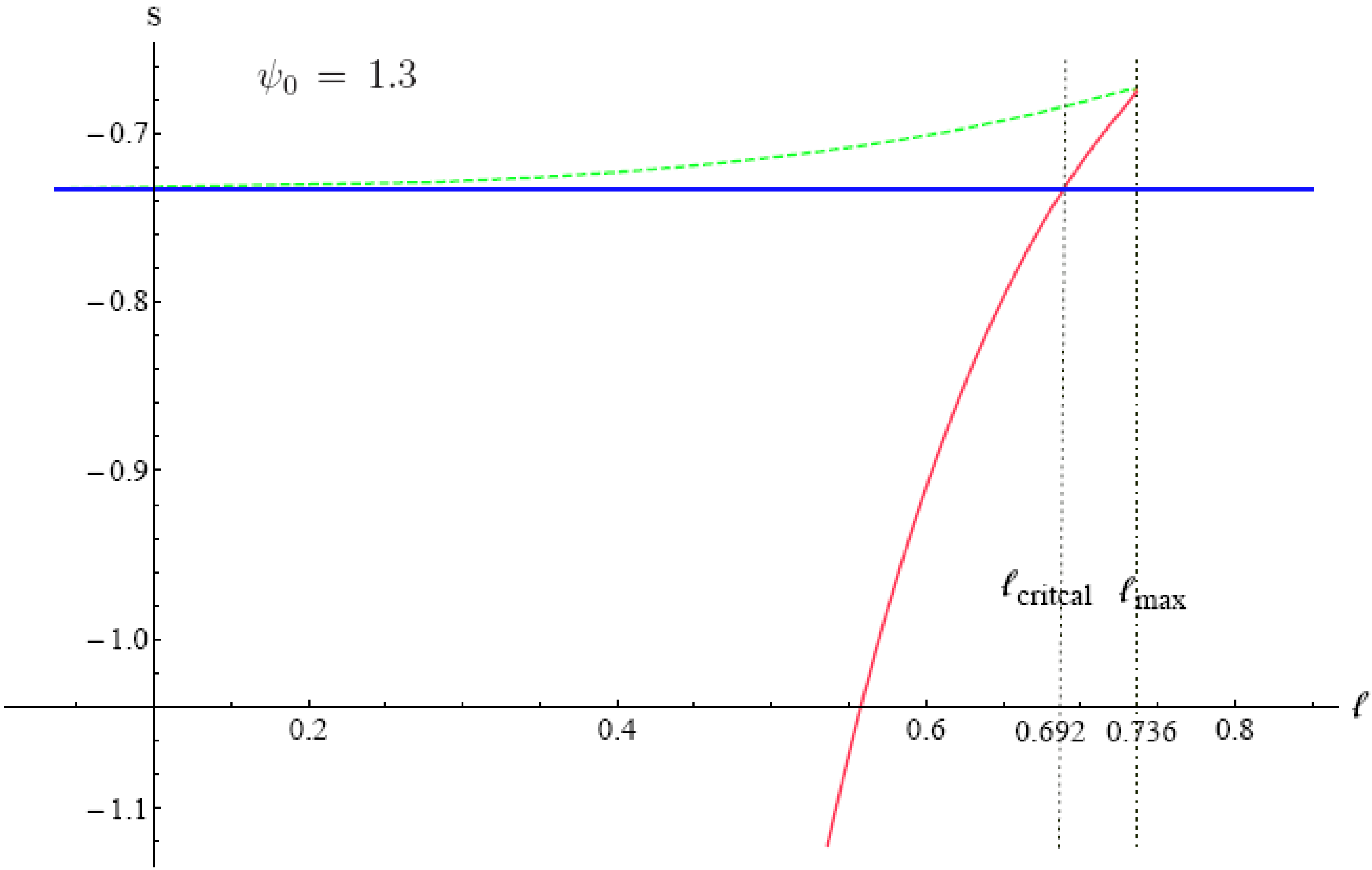
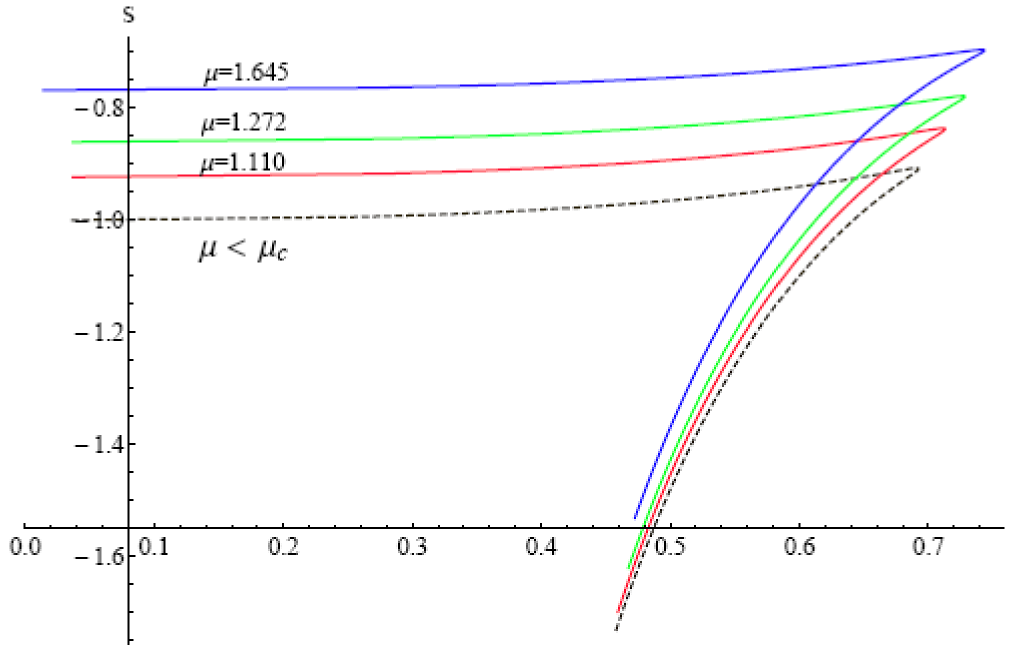
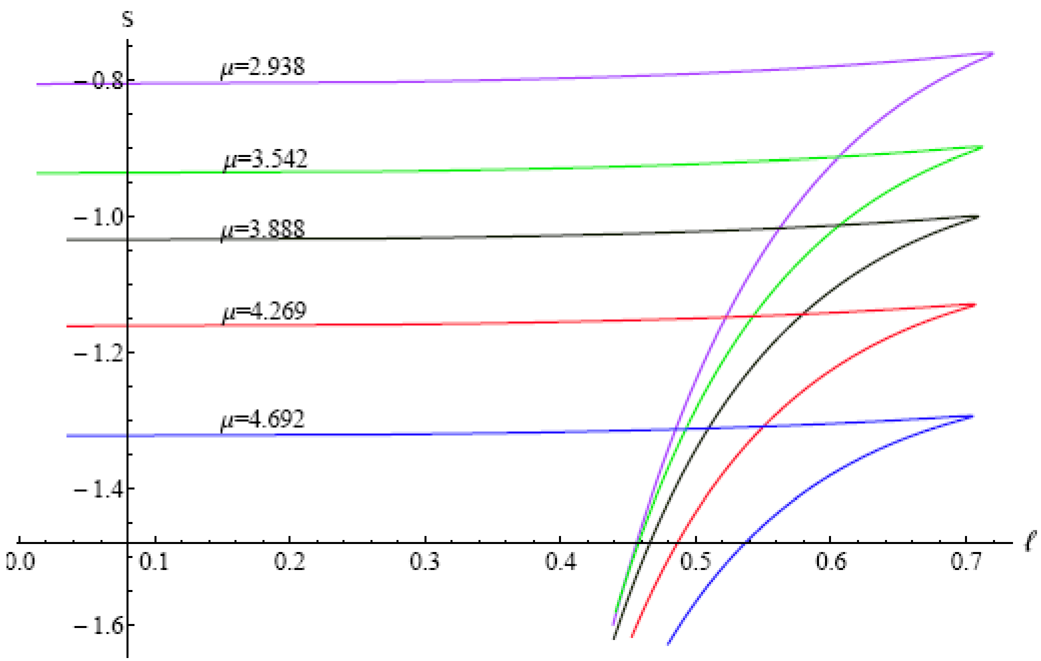


Figure: The dashed green and solid red curves come from the connected solutions, while the solid blue one comes from the disconnected one. The lowest curve is physically favored compared with others because it has minimal entropy.

With a fixed ℓ, the entanglement entropy increases as μ becomes large.



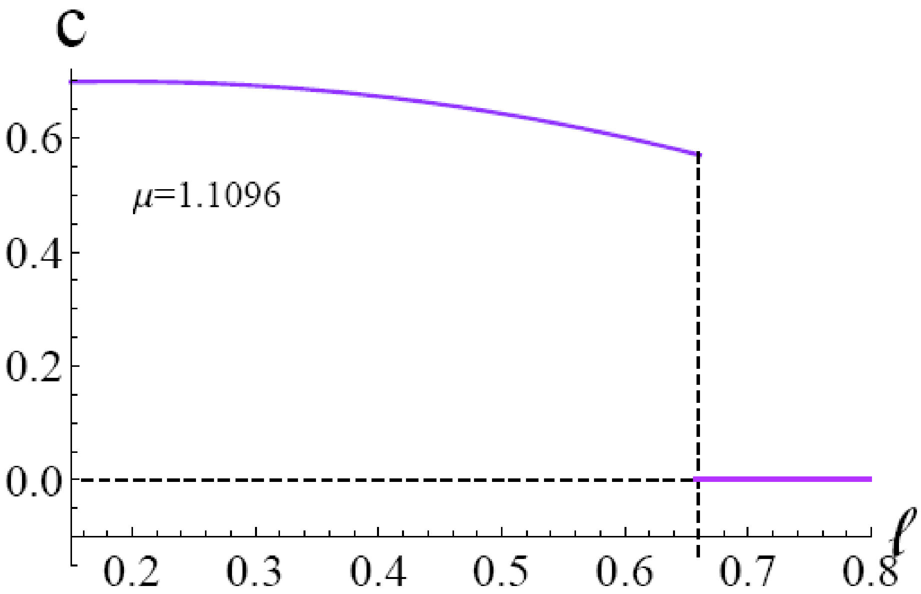


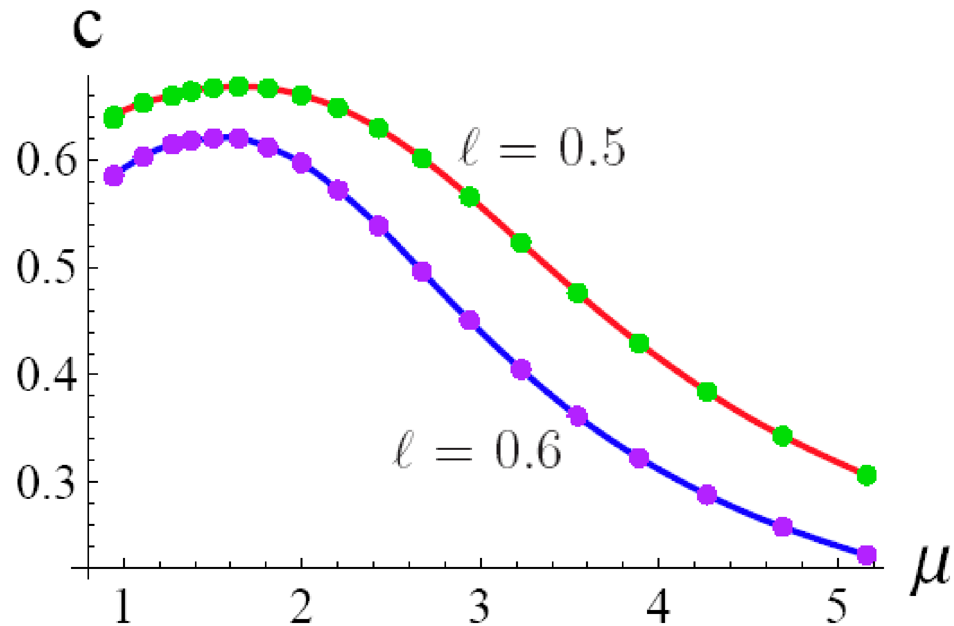
With a fixed ℓ, the entanglement entropy decreases as μ becomes large.

Entropic c-Function

$$c(\ell) = \ell^3 \frac{ds(\ell)}{d\ell}$$

which measure the degrees of freedom at energy scale ~ 1/ℓ





* T. Nishioka and T. Takayanagi, hep-th/0611035

HEE in half embedding case:

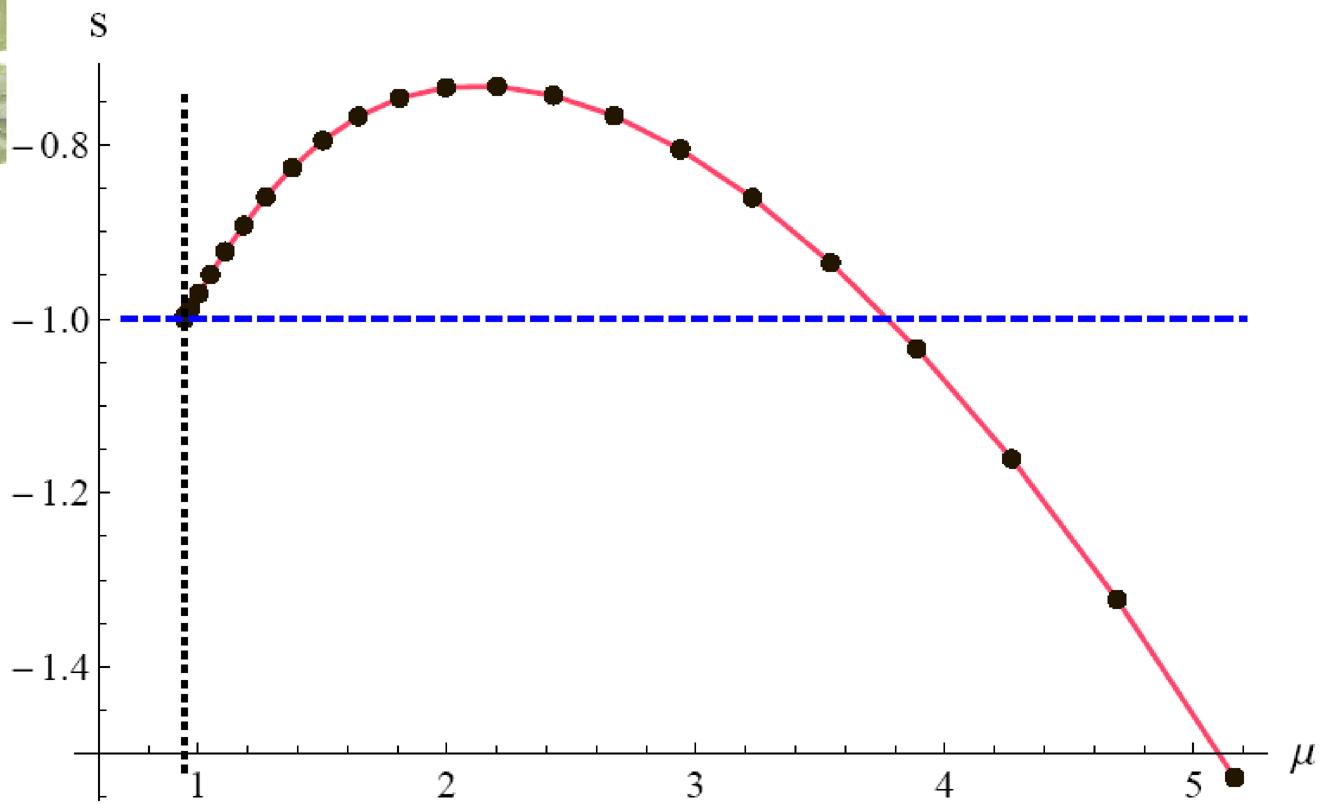


Figure: The solid red curve denotes the entropy in the superconductor phase, while the dashed blue line is for the pure AdS soliton solution. The entropy rises as the chemical potential μ is increased after the phase transition, arrives at its maximum at a certain μ, and then decreases monotonously.

The belt width ℓ does not play the essential role for the non-monotonic behavior of the entanglement entropy !

A clearly physical interpretation is called for.

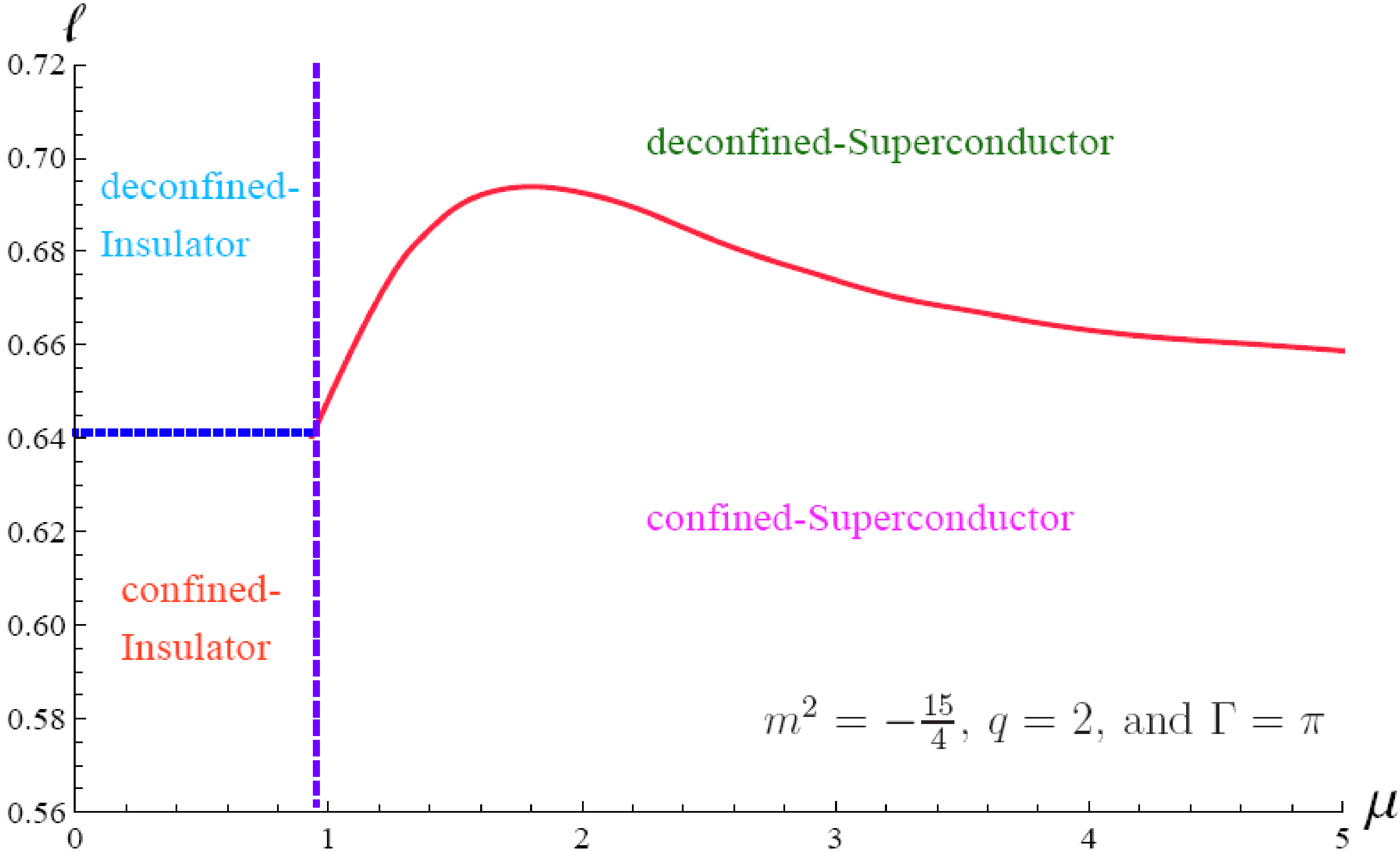


Figure: The phase diagram of entanglement entropy with the belt embedding in the holographic insulator/superconductor model for. The phase boundary between the confining phase and deconfining phase is denoted by the dotted blue line and solid red curve, while the insulator phase and superconductor phase are separated by the vertical dashed line.

P-wave Superconductor Phase Transition

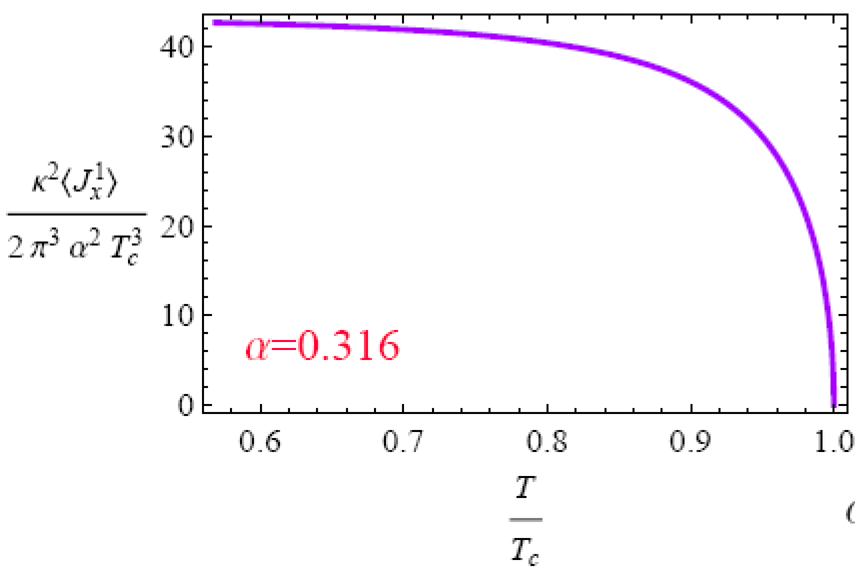
Einstein-Yang-Mills theory in five-dimensional asymptotically AdS spacetime.

$$S = \int d^5x\sqrt{-g}[\frac{1}{2\kappa^2}(R + \frac{12}{L^2}) - \frac{1}{4\hat{g}^2}F^a_{\mu\nu}F^{a\mu\nu}] \quad \text{where} \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}A^b_\mu A^c_\nu$$

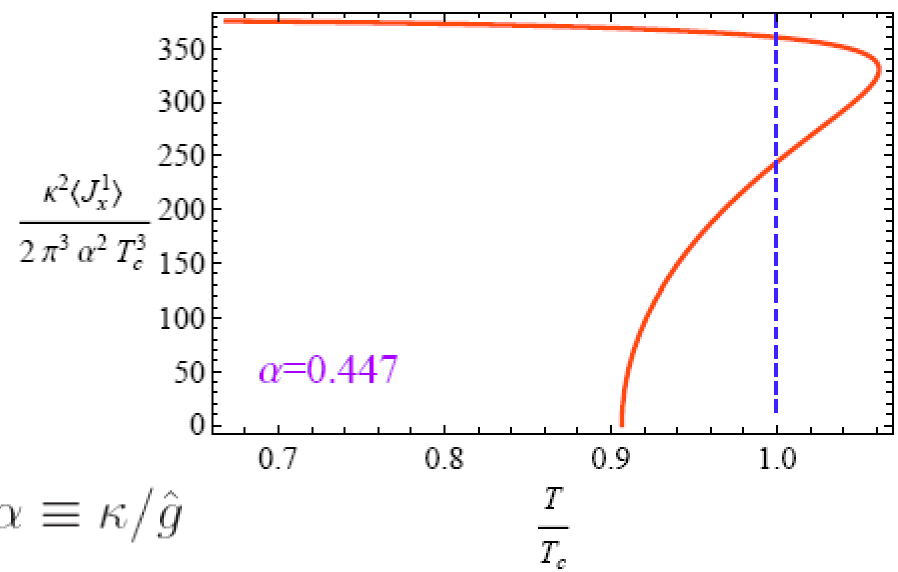
Ansatz for the metric and Yang-Mills field are chosen by

$$ds^2 = -N(r)\sigma(r)^2dt^2 + \frac{1}{N(r)}dr^2 + r^2f(r)^{-4}dx^2 + r^2f(r)^2(dy^2 + dz^2),$$

$$A = \phi(r)\tau^3dt + w(r)\tau^1dx.$$



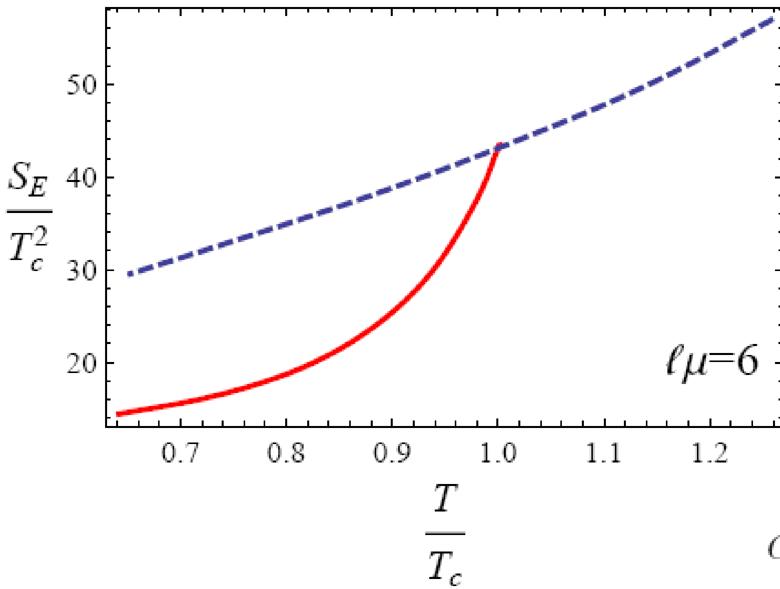
$$\alpha \equiv \kappa/\hat{g}$$

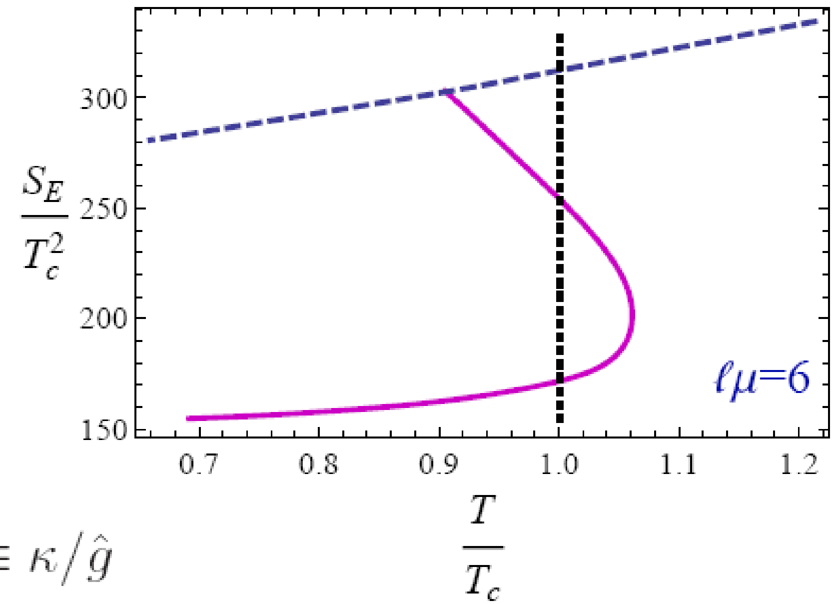


* M. Ammon, J. Erdmenger, V. Grass, P. Kerner , A. O'Bannon, arXiv:0912.3515

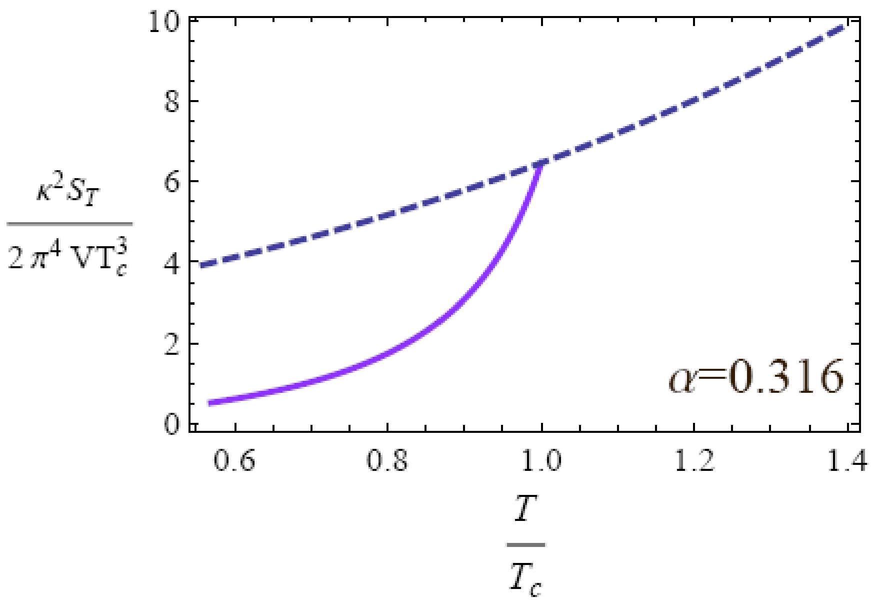
* Rong-Gen Cai, Song He, Li Li, Yun-Long Zhang, arXiv:1204.5962

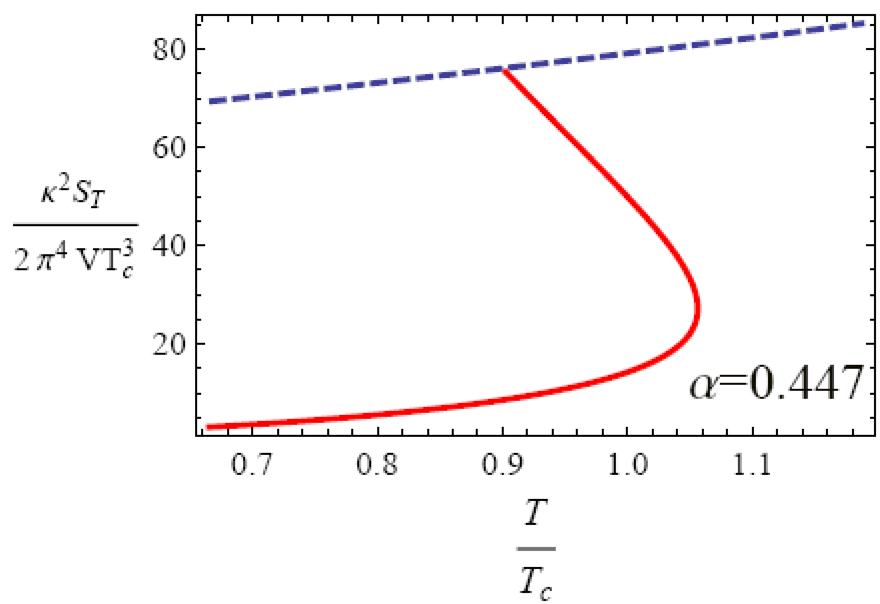
Entanglement Entropy and Thermal Entropy





$$\alpha \equiv \kappa / \hat{g}$$



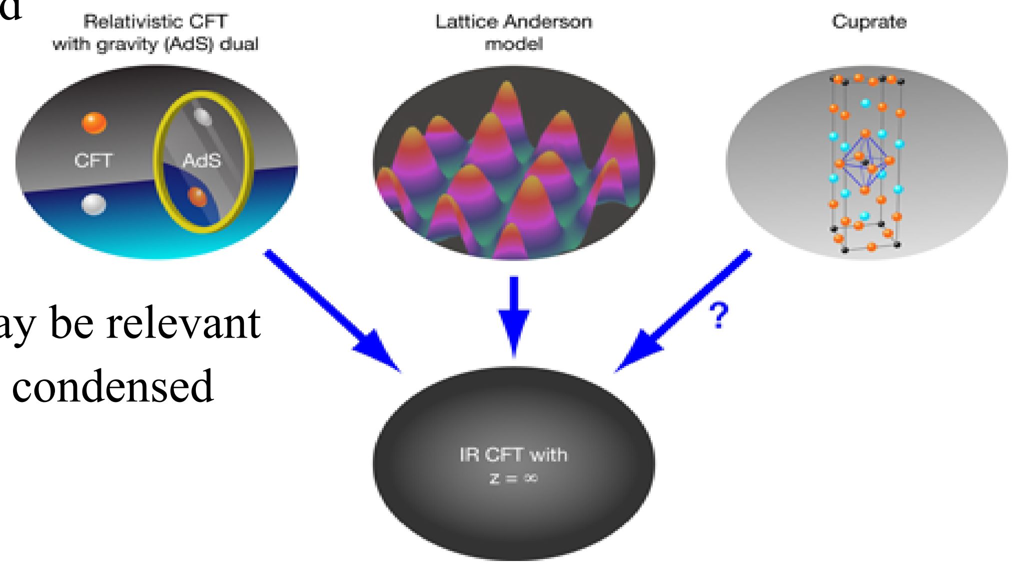


Conclusions

- Tried to convince that AdS/CFT is a useful tool for studying strongly interacting field theories:

    equations of state,
    correlation functions,
    transport properties……

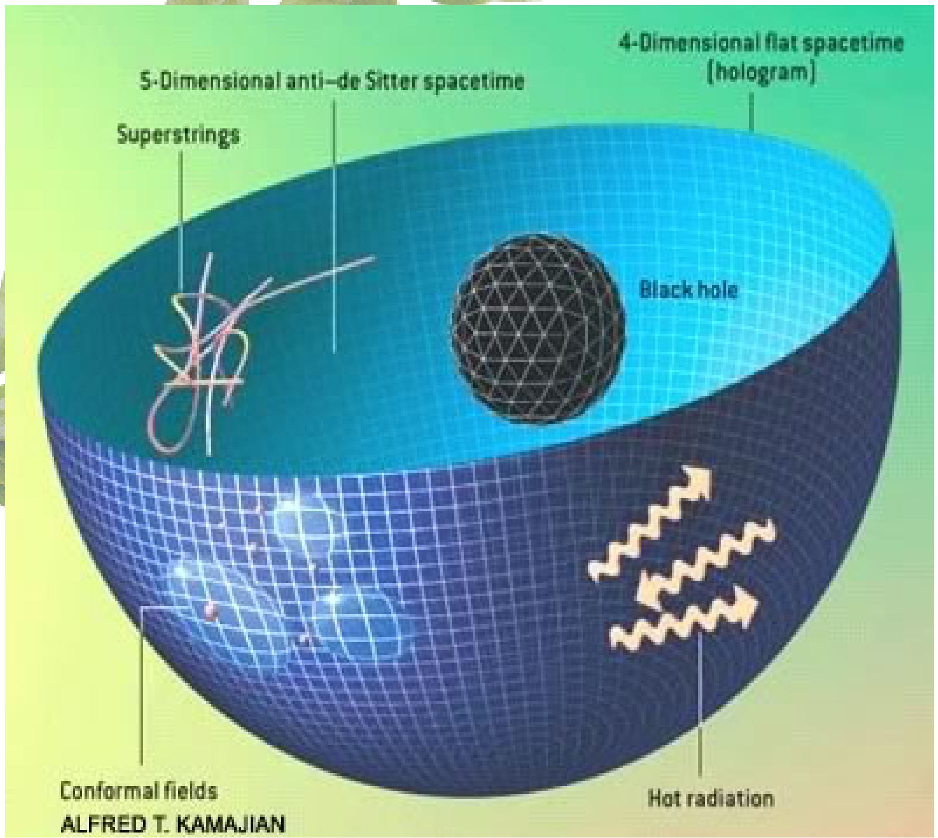
The hope is that this holographic method may be relevant for understanding real world strong coupled condensed matter systems.

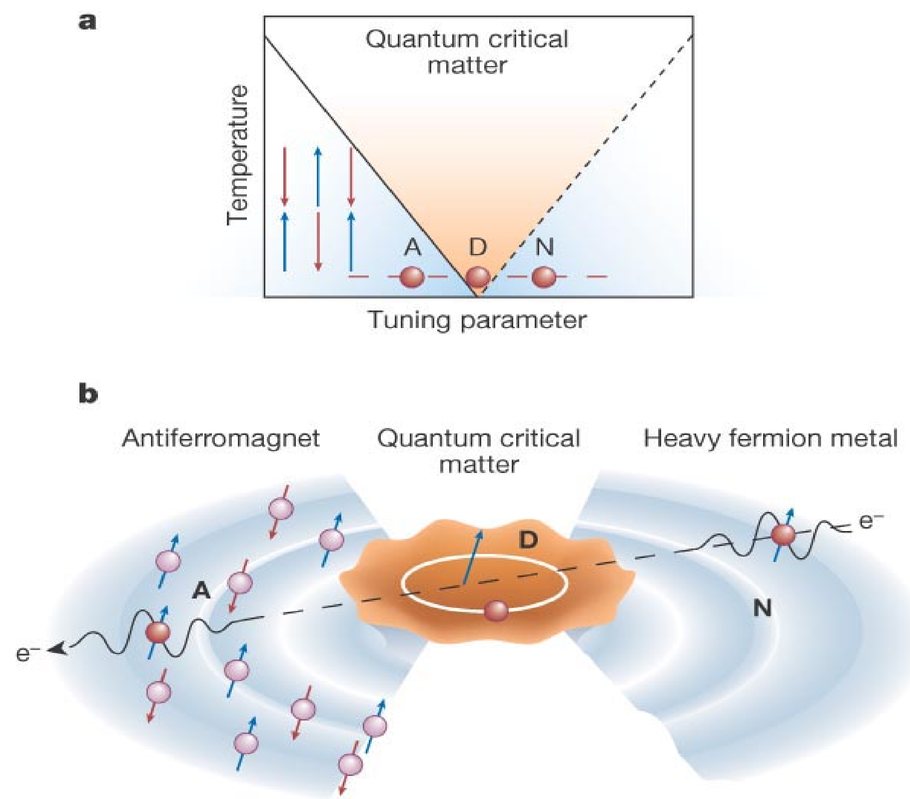


- The entanglement entropy (EE) is a useful bridge between gravity (string theory) and condensed matter physics.

Gravity ⟷ Entanglement ⟷ Cond-mat. systems

$g_{\mu\nu}$ $\quad S_A \approx \text{Area}$ $\quad |\Psi\rangle$





Just as Horowitz and Polchinski wrote: We find it difficult to believe that nature does not make use of it, but the precise way in which it does so remains to be discovered。

Thank you !