Quantum Field Theory
Chapter 1, Homework & Solution

1. Show that the combination

$$\frac{d^3p}{2E}, \qquad \text{with} \quad E = \sqrt{\vec{p}^{\,2} + m^2}$$

which occurs frequently in phase space calculation integration is invariant under Lorentz transformation.

Solution: Under the Lorentz transformation in z-direction, we have the relations

$$p'_z = \gamma\left(p_z - \beta E\right)$$
$$E' = \gamma\left(E - \beta p_z\right)$$

Then

$$dp'_z = \gamma\left(dp_z - \beta dE\right)$$

From energy momentum relation, we get

$$EdE = p_z dp_z$$

and

$$dp'_z = \gamma\left(dp_z - \beta\frac{p_z}{E}dp_z\right) = \gamma\frac{dp_z}{E}\left(E - \beta p_z\right)$$

Hence

$$\frac{dp'_z}{E'} = \frac{dp_z}{E}$$

and

$$\frac{d^3p'}{2E'} = \frac{d^3p}{2E}$$

Alternatively, we can make use of the identity,

$$\int dp_0\delta\left(p_0^2 - (\vec{p}^{\,2} + m^2\right)\theta\left(p_0\right) = \frac{1}{2\sqrt{\vec{p}^{\,2} + m^2}} = \frac{1}{2E}$$

to write

$$\frac{d^3p}{2E} = d^4p\delta\left(p^2 - m^2\right)$$

which is clearly Lorentz invariant.

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2. Consider a system where 2 particles interacting with eac other through potential energy $V\left(\vec{x}_1 - \vec{x}_2\right)$ so that the Lagrangian is of the form,

$$L = \frac{m_1}{2}\left(\frac{d\vec{x}_1}{dt}\right)^2 + \frac{m_2}{2}\left(\frac{d\vec{x}_2}{dt}\right)^2 - V\left(\vec{x}_1 - \vec{x}_2\right)$$

   (a) Show that this Lagrangian is invariant under the spatial translation given by

$$\vec{x}_1 \to \vec{x}'_1 = \vec{x}_1 + \vec{a}, \qquad \vec{x}_2 \to \vec{x}'_2 = \vec{x}_2 + \vec{a},$$

   where $\vec{a}$ is an arbitrary vector.

   (b) Use Noether's theorem to construct the conserved quantity corresponding to this symmetry.

Solution :
a) It is obvious that from

$$\frac{d\vec{x}'_1}{dt} = \frac{d\vec{x}_1}{dt}, \qquad \frac{d\vec{x}'_2}{dt} = \frac{d\vec{x}_2}{dt}$$
$$\vec{x}'_1 - \vec{x}'_2 = \vec{x}_1 - \vec{x}_2$$

that L is invariant under translation.
b) For infinitesmal translation

$$\delta\vec{x}_1 = \vec{a}, \qquad \delta\vec{x}_2 = \vec{a}$$

From Noether's theorem, the conserved charge is

$$J_i a_i = \frac{\partial L}{\partial\left(\partial_0 x_{1j}\right)}\delta x_{1j} + \frac{\partial L}{\partial\left(\partial_0 x_{2j}\right)}\delta x_{2j} = m_1\partial_0 x_{1j}a_j + m_2\partial_0 x_{2j}a_j$$

Or

$$J_i = m_1\partial_0 x_{1i} + m_2\partial_0 x_{2i}$$

This is the usual total momentum of this 2 particle system.

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3. Compute the following physical quantities in the right units.

   (a) The total cross section for $e^+e^- \to \mu^+\mu^-$ at high energies is of the form,

   $$\sigma\left(e^+e^- \to \mu^+\mu^-\right) = \frac{4\pi\alpha^2}{3s}, \qquad s = 4E^2, \qquad E\ \text{:energy of } e^- \text{ in cm frame}, \ \ \alpha \text{ fine structure constant}$$

   Compute the cross section for the energies $E = 100Gev$, $7Tev$

   (b) The formula for the μ decay is given by

   $$\Gamma\left(\mu \to e\nu\bar{\nu}\right) = \frac{G_F^2 M_\mu^3}{192\pi^3}, \qquad G_F \text{ is the Fermi constant}, \qquad M_\mu \text{ the proton mass}$$

   Compute the muon lifetime in seconds.

Solution :
a) $E = 100Gev$

$$\sigma = \frac{4\pi\alpha^2}{3s} = \frac{4\times 3.14}{3\times 4\times (100Gev)^2}\left(\frac{1}{137}\right)^2 = 0.557\times 10^{-8}Gev^{-2}$$

Use $\hbar c = 1.973\times 10^{-11}Mev - cm$ we get

$$\sigma(100Gev) = 0.557\times 10^{-8}Gev^{-2}\times(1.973\times 10^{-11}Mev - cm)^2 = 3.89\times 10^{-36}cm^2$$

$E = 7Tev$

$$\sigma\left(7Tev\right) = \sigma(100Gev)\times\left(\frac{100Gev}{7Tev}\right)^2 = 3.89\times\left(\frac{1}{70}\right)^2 = 8.4\times 10^{-40}cm^2$$

b)

$$\Gamma\left(\mu \to e\nu\bar{\nu}\right) = \frac{G_F^2 m_\mu^5}{192\pi^3} = \frac{\left(1.166\times 10^{-5}Gev^{-2}\right)^2(105Mev)^5}{192\times(3.14)^3} = 2.9\times 10^{-19}Gev$$

Use $\hbar = 6.58\times 10^{-22}Mev - sec$

$$\Gamma = 2.9\times 10^{-19}Gev\times\left(6.58\times 10^{-22}Mev - sec\right)^{-1} = 4.4\times 10^5\ \text{sec}^{-1}$$

$\Longrightarrow$

$$1/\Gamma = 2.2\times 10^{-6}\ \text{sec}$$

4. Construct the Lorentz transformation for motion of coordinate axis in arbitrary direction by using the fact that coordinates perpendicualr to the direction of motion remain unchanged.

Solution :
Let $\vec{v}$ be the velocity of the coordinate system $\vec{x}$. We can decompose $\vec{x}$ as

$$\vec{x} = \vec{x}_\perp + \vec{x}_\parallel, \qquad \text{with} \quad \vec{x}_\parallel = (\vec{x}\cdot\hat{v})\hat{v}, \qquad \vec{x}_\perp = \vec{x} - \vec{x}_\parallel = \vec{x} - (\vec{x}\cdot\hat{v})\hat{v}$$

Under the Lorentz transformation with $\vec{v}$, we have

$$x'_\parallel = \gamma\left(x_\parallel - vx_0\right), \qquad x'_0 = \gamma(x_0 - vx_\parallel), \qquad \vec{x}'_\perp = \vec{x}_\perp$$

Or

$$(\vec{x}' \cdot \hat{v}) = \gamma[(\vec{x} \cdot \hat{v}) - vx_0], \qquad x'_0 = \gamma[x_0 - v(\vec{x} \cdot \hat{v})], \qquad \vec{x}'_\perp = \vec{x}_\perp$$

These can be written as

$$\begin{aligned} \vec{x}' &= \vec{x}'_\perp + \vec{x}'_\parallel = (\vec{x}' \cdot \hat{v})\hat{v} + \vec{x}'_\perp = \gamma[(\vec{x} \cdot \hat{v}) - vx_0]\hat{v} + \vec{x} - (\vec{x} \cdot \hat{v})\hat{v} \\ &= \vec{x} - vx_0\hat{v} + (\gamma - 1)(\vec{x} \cdot \hat{v})\hat{v} \end{aligned}$$

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5. Electric and magnetic fields, $\vec{E}$, $\vec{B}$, combine into an antisymmetric second rank tensor unde the Lorentz transformtion,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad \text{with} \quad F^{0i} = \partial^0 A^i - \partial^i A^0 = -E^i, \quad F^{ij} = \partial^i A^j - \partial^j A^i = -\epsilon_{ijk} B_k$$

These Minkowski tensors have the following property under the Lorentz transformation,

$$F^{\mu\nu} \to F'^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta}, \qquad \Lambda^\mu_\alpha : \text{ matrix element of Lorentz transformation}$$

Suppose an inertial frame O' moves with respect to O with velocity v in the positive x-direction.

(a) Find the relations between the electric and magnetic fields, $\vec{E}'$, $\vec{B}'$,in the O' and those in the O frame.

(b) Show that the combination $\vec{E} \cdot \vec{B}$, does not change from O to O' frames.

(c) Show that the combination $\vec{E}^2 - \vec{B}^2$, does not change either.

Solution:
a) For convenience write the Lorentz transformation as

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{with} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Then we get the transformation relation

$$E^{1\prime} = F'^{10} = \Lambda^1_1 \Lambda^0_0 F^{10} + \Lambda^1_0 \Lambda^0_1 F^{01} = (\gamma^2 - \beta^2\gamma^2)E^1 = E^1$$

$$E^{2\prime} = F'^{20} = \Lambda^2_2 \Lambda^0_0 F^{20} + \Lambda^2_2 \Lambda^0_1 F^{21} = \left(\gamma E^2 - \beta\gamma\left(-B^3\right)\right) = \gamma E^2 + \beta\gamma B^3$$

$$E^{3\prime} = F'^{30} = \Lambda^3_3 \Lambda^0_0 F^{30} + \Lambda^3_3 \Lambda^0_1 F^{31} = (\gamma E^3 - \beta\gamma B^2) = \gamma E^3 - \beta\gamma B^2$$

$$B^{1\prime} = F'^{23} = \Lambda^2_2 \Lambda^3_3 F^{23} = B^1$$

$$B^{2\prime} = F'^{31} = \Lambda^3_3 \Lambda^1_1 F^{31} + \Lambda^3_3 \Lambda^1_0 F^{30} = (\gamma B^2 - \beta\gamma E^3)$$

$$B^{3\prime} = F'^{12} = \Lambda^1_1 \Lambda^2_2 F^{12} + \Lambda^1_0 \Lambda^2_2 F^{02} = (\gamma B^3 - \beta\gamma(-E^2)) = \gamma B^3 + \beta\gamma E^2$$

b)

$$\begin{aligned} \vec{B}' \cdot \vec{E}' &= E^1 B^1 + \left(\gamma E^2 + \beta\gamma B^3\right)\left(\gamma B^2 - \beta\gamma E^3\right) + \left(\gamma E^3 - \beta\gamma B^2\right)\left(\gamma B^3 + \beta\gamma E^2\right) \\ &= E^1 B^1 + E^2 B^2 + E^3 B^3 = \vec{B} \cdot \vec{E} \end{aligned}$$

c)

$$\begin{aligned} \vec{E}'^2 - \vec{B}'^2 &= \left(E^1\right)^2 + \left(\gamma E^2 + \beta\gamma B^3\right)^2 + \left(\gamma E^3 - \beta\gamma B^2\right)^2 - \left(B^1\right)^2 - \left(\gamma B^2 - \beta\gamma E^3\right)^2 - \left(\gamma B^3 + \beta\gamma E^2\right)^2 \\ &= \left(E^1\right)^2 + \left(E^2\right)^2 + \left(E^3\right)^2 - \left(B^1\right)^2 - \left(B^2\right)^2 - \left(B^3\right)^2 = \vec{E}^2 - \vec{B}^2 \end{aligned}$$

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Quantum Field Theory
Homework set 2, Solution

1. The Dirac Hamiltonian for free particle is given by

$$H = \vec{\alpha} \cdot \vec{p} + \beta m$$

The angular momentum operator is of the form,

$$\vec{L} = \vec{r} \times \vec{p}$$

(a) Compute the commutators,

$$\left[\vec{L},\ H\right]$$

Is $\vec{L}$ conserved?

(b) Define $\vec{S} = -\frac{i}{4}\left(\vec{\alpha} \times \vec{\alpha}\right)$ and show that

$$\left[\vec{L} + \vec{S},\ H\right] = 0$$

(c) Show that $\vec{S}$ satisfy the angular momentum algebra, i.e.

$$[S_i, S_j] = i\varepsilon_{ijk} S_k$$

and

$$\vec{S}^2 = \frac{3}{4}.$$

Solution :

a)

$$[L_1, \vec{\alpha} \cdot \vec{p} + \beta m] = [x_2 p_3 - x_3 p_2,\quad \alpha_2 p_2 + \alpha_3 p_3] = -i\alpha_2 p_3 - (-i\alpha_3 p_2) = -i\left(\vec{\alpha} \times \vec{p}\right)_1$$

We can generalize this to

$$\left[\vec{L},\quad \vec{\alpha} \cdot \vec{p} + \beta m\right] = -i\left(\vec{\alpha} \times \vec{p}\right)$$

So $\vec{L}$ is not conserved.

b)

$$[S_1, H] = -\frac{i}{2}\left[\alpha_2\alpha_3,\ \vec{\alpha} \cdot \vec{p} + \beta m\right]$$

It is easy to verify that

$$[\alpha_2\alpha_3, \alpha_1] = \alpha_2\alpha_3\alpha_1 - \alpha_1\alpha_2\alpha_3 = 0,$$

$$[\alpha_2\alpha_3, \alpha_2] = \alpha_2\alpha_3\alpha_2 - \alpha_2\alpha_2\alpha_3 = -2\alpha_3,$$

Similarly,

$$[\alpha_2\alpha_3, \alpha_3] = 2\alpha_2, \qquad [\alpha_2\alpha_3, \beta] = 0$$

Then

$$[S_1, H] = -\frac{i}{2}2\left(-\vec{\alpha} \times \vec{p}\right)_1$$

Or

$$[\vec{S}, H] = i\vec{\alpha} \times \vec{p}$$

So

$$\left[\vec{L} + \vec{S},\quad H\right] = 0$$

i.e. the total angular momenta is conserved.

c)

$$[S_1, S_2] = \left(\frac{-i}{2}\right)^2 [\alpha_2\alpha_3,\ \alpha_3\alpha_1] = -\frac{1}{4}\left(\alpha_2\alpha_3\alpha_3\alpha_1 - \alpha_3\alpha_1\alpha_2\alpha_3\right) = \frac{1}{2}\alpha_1\alpha_2 = iS_3$$

$$S_1^2 = \left(\frac{-i}{2}\right)^2 \alpha_2\alpha_3\alpha_2\alpha_3 = \frac{1}{4}$$

Then

$$\vec{S}^2 = \frac{3}{4}, \qquad \Longrightarrow \qquad S = \frac{1}{2}$$

2. The Dirac spinors are of the form,

$$u(p,s) = \sqrt{E+m}\begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_s, \qquad v(p,s) = \sqrt{E+m}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\chi_s \qquad s = 1,2$$

where

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(a) Show that

$$\bar{u}(p,s)\,u(p,s') = 2m\delta_{ss'}, \qquad \bar{v}(p,s)\,v(p,s') = -2m\delta_{ss'}$$
$$\bar{v}(p,s)\,u(p,s') = 0, \qquad \bar{u}(p,s)\,v(p,s') = 0$$
$$v^\dagger(-p,s)\,u(p,s') = 0, \qquad u^\dagger(p,s)\,v(-p,s') = 0$$

(b) Show that

$$\sum_s u_\alpha(p,s)\,\bar{u}_\beta(p,s) = (\not{p}+m)_{\alpha\beta}$$
$$\sum_s v_\alpha(p,s)\,\bar{v}_\beta(p,s) = (\not{p}-m)_{\alpha\beta}$$

Solution :
a)

$$\begin{aligned}\bar{u}(p,s)\,u(p,s') &= (E+m)\,\chi_s^\dagger \begin{pmatrix} 1 & \dfrac{-\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_s \\ &= (E+m)\,\chi_s^\dagger\left(1 - \frac{\vec{p}^{\,2}}{(E+m)^2}\right)\chi_s = 2m\delta_{ss'}\end{aligned}$$

$$\begin{aligned}\bar{v}(p,s)\,v(p,s') &= (E+m)\,\chi_s^\dagger \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} & -1 \end{pmatrix}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\chi_{s'} \\ &= (E+m)\,\chi_s^\dagger\left(\frac{\vec{p}^{\,2}}{(E+m)^2} - 1\right)\chi_s = -2m\delta_{ss'}\end{aligned}$$

$$\bar{u}(p,s)\,v(p,s') = (E+m)\,\chi_s^\dagger\begin{pmatrix} 1 & \dfrac{-\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\chi_{s'} = 0$$

$$\bar{v}(p,s)\,u(p,s') = (E+m)\,\chi_s^\dagger\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} & -1 \end{pmatrix}\begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_s = 0$$

$$v^\dagger(-p,s)\,u(p,s') = (E+m)\,\chi_s^\dagger\begin{pmatrix} -\dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_s = 0$$

$$u^\dagger(p,s)\,v(-p,s') = (E+m)\,\chi_s^\dagger\begin{pmatrix} 1 & \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\begin{pmatrix} -\dfrac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\chi_{s'} = 0$$

b)The spin sum for $u-$spinors,

$$\begin{aligned}\sum_s u_\alpha(p,s)\bar{u}_\beta(p,s) &= (E+m)\begin{pmatrix}1\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\end{pmatrix}\sum_s \chi_s\chi_s^\dagger\begin{pmatrix}1 & -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\end{pmatrix} = (E+m)\begin{pmatrix}1 & -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} & \frac{-\vec{p}^2}{(E+m)^2}\end{pmatrix}\\ &= \begin{pmatrix}E+m & -\vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & -E+m\end{pmatrix} = \not{p}+m\end{aligned}$$

where we have used

$$\sum_s \chi_s\chi_s^\dagger = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Similarly for the $v-$spinor,

$$\begin{aligned}\sum_s v_\alpha(p,s)\bar{v}_\beta(p,s) &= (E+m)\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\\ 1\end{pmatrix}\chi_s\chi_s^\dagger\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E+m} & -1\end{pmatrix} = (E+m)\begin{pmatrix}\frac{\vec{p}^2}{(E+m)^2} & -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} & -1\end{pmatrix}\\ &= \begin{pmatrix}E-m & -\vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & -(E+m)\end{pmatrix} = \not{p}-m\end{aligned}$$

3. Suppose a free Dirac particle at t=0, is described by a wavefunction,

$$\psi\left(0,\vec{x}\right) = \frac{1}{(\pi d^2)^{3/4}}\exp\left(-\frac{r^2}{2d^2}\right)\omega$$

where d is some constant and

$$\omega = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

Compute the wavefunction for $t\neq 0$. What happens when d is very small?

Solution :
Expand $\psi\left(0,\vec{x}\right)$ in terms of spinors

$$\psi\left(0,\vec{x}\right) = \sum_s\int\frac{d^3p}{\sqrt{(2\pi)^3 2E_p}}\left[b(p,s)\,u(p,s)\,e^{i\vec{p}\cdot\vec{x}} + d^\dagger(p,s)\,v(p,s)\,e^{-i\vec{p}\cdot\vec{x}}\right]$$

We can compute the expansion coefficients by the orthogonality properties

$$b(p,s) = \int\frac{d^3xe^{-i\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3 2E_p}}u^\dagger(p,s)\,\psi\left(\vec{x},0\right) = \int\frac{d^3xe^{-i\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3 2E_p}}u^\dagger(p,s)\,\omega\frac{1}{(\pi d^2)^{3/4}}\exp\left(-\frac{r^2}{2d^2}\right)$$

The Fourier transform of the Gaussian wave packet can be calculated as follows,

$$\begin{aligned}\int d^3xe^{-i\vec{p}\cdot\vec{x}}\exp\left(-\frac{\vec{x}^2}{2d^2}\right) &= \int d^3x\exp[-\frac{1}{2d^2}(\vec{x}+i\vec{p}d^2)^2 - \frac{\vec{p}^2d^2}{2}]\\ &= \exp(-\frac{\vec{p}^2d^2}{2})\int d^3x\exp[-\frac{1}{2d^2}(\vec{x})^2] = \exp(-\frac{\vec{p}^2d^2}{2})(\sqrt{2\pi}d)^3\end{aligned}$$

where we have used

$$\int dxe^{-x^2} = \sqrt{\pi}$$

$$b(p,s) = \sqrt{E_p+m}\delta_{s1}\exp(-\frac{\vec{p}^2d^2}{2})\frac{1}{\sqrt{2E_p}}\frac{1}{\pi^{3/4}}$$

Similarly

$$d^\dagger(p,s) = \int\frac{d^3xe^{-ip\cdot x}}{\sqrt{(2\pi)^3 2E_p}}v^\dagger(p,s)\,\psi\left(\vec{x},0\right) = \int\frac{d^3xe^{-i\vec{p}\cdot\vec{x}}}{\sqrt{(2\pi)^3 2E_p}}v^\dagger(p,s)\,\omega\frac{1}{(\pi d^2)^{3/4}}\exp\left(-\frac{r^2}{2d^2}\right)$$

$$v^{\dagger}(p,s)\,\omega = \sqrt{E+m}\chi_s^{\dagger}\left(\begin{array}{cc} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} & 1 \end{array}\right)\left(\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right) = \chi_s^{\dagger}\left(\begin{array}{c} p_z \\ p_x + ip_y \end{array}\right)$$

and

$$d^{\dagger}(p,s) = \sqrt{E_p+m}\exp(-\frac{\vec{p}^2 d^2}{2})\frac{1}{\sqrt{2E_p}}\frac{1}{\pi^{3/4}}\chi_s^{\dagger}\left(\begin{array}{c} p_z \\ p_x + ip_y \end{array}\right)\frac{1}{E+m}$$

For non-zero t we get

$$\psi\left(t,\vec{x}\right) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3\,2E_p}}\left[b(p,s)\,u(p,s)\,e^{-iE_pt}e^{i\vec{p}\cdot\vec{x}} + d^{\dagger}(p,s)\,v(p,s)\,e^{iE_pt}e^{-i\vec{p}\cdot\vec{x}}\right]$$

Note that

$$\left|\frac{d^{\dagger}(p,s)}{b(p,s)}\right| \sim \frac{p}{E+m}$$

This shows that the negative energy amplitude becomes appreciable when $p \sim m$.

4. Consider a 2×2 hermitian matrix defined by

$$X = x_0 + \vec{\sigma}\cdot\vec{x}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices and $\left(x_0, \vec{x}\right)$ are space-time coordinates.

(a) Compute the determinant of X

(b) Suppose U is a 2×2 matrix with $\det U = 1$. Define a new 2×2 matrix by a similarity transformation,

$$X' = UXU^{\dagger}$$

Show that X' can be written as

$$X' = x_0' + \vec{\sigma}\cdot\vec{x}'$$

(c) Show that the relation between $\left(x_0, \vec{x}\right)$ and $\left(x_0', \vec{x}'\right)$ is a Lorentz transformation.

(d) Suppose U is of the form,

$$U = \left(\begin{array}{cc} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{array}\right)$$

Find the relation between $\left(x_0, \vec{x}\right)$ and $\left(x_0', \vec{x}'\right)$.

Solution:

a)

$$X = \left(\begin{array}{cc} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{array}\right)$$

$$\det X = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

b) Note that X is Hermitian,

$$X^{\dagger} = \left(x_0 + \vec{\sigma}\cdot\vec{x}\right)^{\dagger} = x_0 + \vec{\sigma}\cdot\vec{x}$$

So is X'

$$X'^{\dagger} = \left(UXU^{\dagger}\right)^{\dagger} = UX^{\dagger}U^{\dagger} = UXU^{\dagger} = X'$$

Expand X' in terms of complete set of 2×2 Hermitian matrices,

$$X' = x_0' + \vec{\sigma}\cdot\vec{x}'$$

c)From the invariance of the determinant

$$\det X' = \det(UXU^{\dagger}) = \det U\,(\det X)\det U^{\dagger} = \det X$$

we see that

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 = x_0'^2 - x_1'^2 - x_2'^2 - x_3'^2$$

So the relation between $\left(x_0, \vec{x}\right)$ and $\left(x_0', \vec{x}'\right)$ is a Lorentz transformation.
d)

$$\begin{aligned} X' &= UXU^\dagger = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \begin{pmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{pmatrix} \begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \\ &= \begin{pmatrix} (x_0+x_3) & e^{2(i\alpha)}(x_1-ix_2) \\ e^{2(-i\alpha)}(x_1+ix_2) & (x_0-x_3) \end{pmatrix} \end{aligned}$$

This correspond to a rotation $z-axis$.
Note that U is not necessarily unitary. In fact the Lorentz boost correspond to

$$U == \begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix}$$

which gives

$$\begin{aligned} UXU^\dagger &= \begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix} \begin{pmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{pmatrix} \begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix} \\ &= \begin{pmatrix} e^{2\alpha}(x_0+x_3) & (x_1-ix_2) \\ (x_1+ix_2) & e^{-2\alpha}(x_0-x_3) \end{pmatrix} \end{aligned}$$

This implies

$$x_0^{'} = \cosh 2\alpha \; x_0 + \sinh 2\alpha \; x_3$$

$$x_3^{'} = \sinh 2\alpha \; x_0 + \cosh 2\alpha \; x$$

For the Lorentz boost along $x-$axis, we can first rotate $\frac{\pi}{2}$ about y-axis using

$$U_y(\beta) = \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \text{for} \quad \beta = \frac{\pi}{2}$$

hen

$$X_2 = U_y^\dagger(\beta) X U_y(\beta) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} x_0+x_1 & -ix_2-x_3 \\ ix_2-x_3 & x_0-x_1 \end{pmatrix}$$

and

$$\begin{aligned} X_3 &= U^\dagger X_2 U = \begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix} \begin{pmatrix} x_0+x_1 & -ix_2-x_3 \\ ix_2-x_3 & x_0-x_1 \end{pmatrix} \begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix} \\ &= \begin{pmatrix} e^{2\alpha}(x_0+x_1) & -(ix_2+x_3) \\ -(x_3-ix_2) & e^{-2\alpha}(x_0-x_1) \end{pmatrix} \end{aligned}$$

$$\begin{aligned} X_4 &= U_y^\dagger(-\beta) X_3 U_y(-\beta) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{2\alpha}(x_0+x_1) & -(ix_2+x_3) \\ -(x_3-ix_2) & e^{-2\alpha}(x_0-x_1) \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} x_3 + \frac{1}{2}e^{-2\alpha}(x_0-x_1) + \frac{1}{2}e^{2\alpha}(x_0+x_1) & \frac{1}{2}e^{2\alpha}(x_0+x_1) - \frac{1}{2}e^{-2\alpha}(x_0-x_1) - ix_2 \\ ix_2 - \frac{1}{2}e^{-2\alpha}(x_0-x_1) + \frac{1}{2}e^{2\alpha}(x_0+x_1) & \frac{1}{2}e^{-2\alpha}(x_0-x_1) - x_3 + \frac{1}{2}e^{2\alpha}(x_0+x_1) \end{pmatrix} \end{aligned}$$

$$x_0' = \frac{1}{2}e^{-2\alpha}(x_0-x_1) + \frac{1}{2}e^{2\alpha}(x_0+x_1) = \cosh 2\alpha x_0 + \sinh 2\alpha x_1$$

$$x_3' = x_3, \qquad x_2' = x_2$$

$$x_1' = -\frac{1}{2}e^{-2\alpha}(x_0-x_1) + \frac{1}{2}e^{2\alpha}(x_0+x_1) = \sinh 2\alpha x_0 + \cosh 2\alpha x_1$$

5. Dirac particle in the presence of electromagnetic field satisfies the equation,

$$\left[\gamma^{\mu}\left(i\partial_{\mu}-eA_{\mu}\right)-m\right]\psi\left(x\right)=0$$

Or

$$i\frac{\partial\psi}{\partial t}=\left[\vec{\alpha}\cdot\left(\vec{p}-e\vec{A}\right)+\beta m+e\phi\right]\psi$$

In the non-relativistic limit, we can write

$$\psi\left(x\right)=e^{-imt}\begin{pmatrix}u\\ l\end{pmatrix}$$

Show that the upper component satisfies the equation,

$$i\frac{\partial u}{\partial t}=\left[\frac{1}{2m}\left(\vec{p}-e\vec{A}\right)^{2}-\frac{e}{2m}\vec{\sigma}\cdot\vec{B}+e\phi\right]u$$

For the case of weak uniform magnetic field $\vec{B}$ we can take $\vec{A}=\frac{1}{2}\vec{B}\times\vec{r}$. Show that

$$i\frac{\partial u}{\partial t}=\left[\frac{1}{2m}\left(\vec{p}\right)^{2}-\frac{e}{2m}\left(\vec{L}+2\vec{S}\right)\cdot\vec{B}\right]u.$$

Solution:
In non-relativistic limit, Dirac equation becomes

$$\begin{pmatrix}i\frac{\partial u}{\partial t}+mu\\ i\frac{\partial l}{\partial t}+ml\end{pmatrix}=\begin{pmatrix}m+e\phi & \vec{\sigma}\cdot\left(\vec{p}-e\vec{A}\right)\\ \vec{\sigma}\cdot\left(\vec{p}-e\vec{A}\right) & -m+e\phi\end{pmatrix}\begin{pmatrix}u\\ l\end{pmatrix}$$

Or

$$i\frac{\partial u}{\partial t}=\left(e\phi\right)u+\vec{\sigma}\cdot\left(\vec{p}-e\vec{A}\right)l$$

$$i\frac{\partial l}{\partial t}=\vec{\sigma}\cdot\left(\vec{p}-e\vec{A}\right)u+\left(-2m+e\phi\right)l$$

From the 2nd equation, we get

$$l=\frac{1}{2m}\vec{\sigma}\cdot\left(\vec{p}-e\vec{A}\right)u$$

where we have neglected $e\phi$ and $\frac{\partial l}{\partial t}$.Substitute this into first equation we get

$$i\frac{\partial u}{\partial t}=\left(e\phi\right)u+\frac{1}{2m}\left[\vec{\sigma}\cdot\left(\vec{p}-e\vec{A}\right)\right]^{2}u$$

Using the identity

$$\left(\vec{\sigma}\cdot\vec{A}\right)\left(\vec{\sigma}\cdot\vec{B}\right)=\vec{A}\cdot\vec{B}+i\vec{\sigma}\cdot\left(\vec{A}\times\vec{B}\right)$$

we get

$$\left[\vec{\sigma}\cdot\left(\vec{p}-e\vec{A}\right)\right]^{2}=\left(\vec{p}-e\vec{A}\right)^{2}+i\vec{\sigma}\cdot\left(\vec{p}-e\vec{A}\right)\times\left(\vec{p}-e\vec{A}\right)$$

Since $\vec{p}$ and $e\vec{A}$ we get

$$\left(\vec{p}-e\vec{A}\right)\times\left(\vec{p}-e\vec{A}\right)=-e\left(\vec{A}\times\vec{p}+\vec{p}\times\vec{A}\right)=+ie\vec{\nabla}\times\vec{A}=ie\vec{B}$$

Then the equation becomes

$$i\frac{\partial u}{\partial t}=\left[\frac{1}{2m}\left(\vec{p}-e\vec{A}\right)^{2}-\frac{e}{2m}\vec{\sigma}\cdot\vec{B}+e\phi\right]u$$

For weak field, we get

$$\begin{aligned}\left(\vec{p}-e\vec{A}\right)^2 &= p^2-e\left(\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p}\right)=p^2-e\left(-i\vec{\nabla}\cdot\frac{1}{2}\vec{r}\times\vec{B}\right)\\ &= p^2-e\left(\vec{B}\cdot\frac{1}{2}\vec{p}\times\vec{r}\right)=p^2-\frac{1}{2}e\left(\vec{B}\cdot\vec{L}\right)\end{aligned}$$

and

$$i\frac{\partial u}{\partial t}=\left[\frac{1}{2m}\left(\vec{p}\right)^2-\frac{e}{2m}\left(\vec{L}+2\vec{S}\right)\cdot\vec{B}\right]u.$$

6. $a_1^\dagger, a_2^\dagger, a_1, a_2$ are creation and annihilation operators satifying the commutation relations

$$\left[a_i, a_j^\dagger\right]=\delta_{ij},\qquad [a_i,a_j]=0,\qquad i,j=1,2$$

Define

$$J_+=a_1^\dagger a_2,\qquad J_-=(J_+)^\dagger,\qquad J_3=\frac{1}{2}\left(a_1^\dagger a_1-a_2^\dagger a_2\right)$$

(a) Compute the commutators

$$[J_x,J_y],\qquad [J_y,J_z],\qquad [J_z,J_x]$$

where $J_x\equiv\frac{1}{2}(J_++J_-)$, $\quad J_y\equiv\frac{1}{2i}(J_+-J_-)$

(b) Define the state $|0\rangle$ by

$$a_i|0\rangle=0,\qquad \text{for}\ \ i=1,2$$

Let the state $|n_1,n_2\rangle$ be

$$|n_1,n_2\rangle=\frac{1}{\sqrt{n_1!n_2!}}\left(a_1^\dagger\right)^{n_1}\left(a_2^\dagger\right)^{n_2}|0\rangle$$

Show that this state is an eigenstate of J_3 and compute the eigenvalue.

(c) Show that this is also eigen state of $J^2=J_1^2+J_2^2+J_3^2$ and compute the eigenvalue.

(d) Show that the state $J_+|n_1,n_2\rangle$ is an eigenstate of J_3. What is the eigenvalue?

———————————————a)

$$[J_+,J_3]=\frac{1}{2}\left[a_1^\dagger a_2,\left(a_1^\dagger a_1-a_2^\dagger a_2\right)\right]=\frac{1}{2}\left(-2a_1^\dagger a_2\right)=-J_+$$

Similarly

$$[J_-,J_3]=J_-$$

Also

$$[J_+,J_-]=[a_1^\dagger a_2,a_2^\dagger a_1]=\left(a_1^\dagger a_1-a_2^\dagger a_2\right)=2J_3$$

Then

$$[J_1,J_3]=\frac{1}{2}[J_++J_-,J_3]=\frac{1}{2}(-J_++J_-)=\frac{1}{2}(-2iJ_2)=-iJ_2$$

and

$$[J_2,J_3]=\frac{1}{2i}[J_+-J_-,J_3]=\frac{1}{2i}(-J_+-J_-)=\frac{1}{2i}(-2J_1)=iJ_1$$

The other commutator is

$$[J_1,J_2]=\frac{1}{4i}[J_++J_-,J_+-J_-]=\frac{1}{4i}(-2\times 2J_3)=iJ_3$$

Define the number operators

$$N_1=a_1^\dagger a_1,\qquad N_2=a_2^\dagger a_2$$

Then we can derive

$$\left[N_1,a_1^\dagger\right]=\left[a_1^\dagger a_1,a_1^\dagger\right]=a_1^\dagger,\cdots\qquad \left[N_1,\left(a_1^\dagger\right)^n\right]=n\left(a_1^\dagger\right)^n$$

and

$$\left[N_2,\left(a_2^\dagger\right)^n\right]=n\left(a_2^\dagger\right)^n$$

From this we see that

$$N_1 \left| n_1, n_2 \right\rangle = \frac{1}{\sqrt{n_1! n_2!}} N_1 \left(a_1^\dagger \right)^{n_1} \left(a_2^\dagger \right)^{n_2} \left| 0 \right\rangle = n_1 \left| n_1, n_2 \right\rangle$$

where we have used

$$N_1 \left| 0 \right\rangle = 0$$

Similarly,

$$N_2 \left| n_1, n_2 \right\rangle = n_2 \left| n_1, n_2 \right\rangle$$

Then

$$J_3 \left| n_1, n_2 \right\rangle = (N_1 - N_2) \left| n_1, n_2 \right\rangle = (n_1 - n_2) \left| n_1, n_2 \right\rangle$$

We can write $\vec{J}^2$ as

$$\vec{J}^2 = \frac{1}{2} [J_+ J_- + J_- J_+] + J_3^2$$

We can write J_+, J_- in term of number operators as

$$J_+ J_- = a_1^\dagger a_2 a_2^\dagger a_1 = a_1^\dagger (1 + a_2^\dagger a_2) a_1 = N_1 (1 + N_2)$$

and

$$J_+ J_- = N_2 (1 + N_1)$$

So

$$\begin{aligned} \vec{J}^2 &= \frac{1}{2} [J_+ J_- + J_- J_+] + + J_3^2 = \frac{1}{2} [(N_1 + N_2) + 2 N_1 N_2] + \frac{1}{4} (N_1 - N_2)^2 \\ &= \frac{1}{4} (N_1 + N_2)(N_1 + N_2 + 2) \end{aligned}$$

and

$$\vec{J}^2 \left| n_1, n_2 \right\rangle = \frac{1}{4} (n_1 + n_2)((n_1 + n_2 + 2)) \left| n_1, n_2 \right\rangle$$

This implies that

$$J = \frac{1}{2} (n_1 + n_2)$$

d) From

$$[J_+, J_3] = -J_+$$

we see that

$$J_3 (J_+ \left| n_1, n_2 \right\rangle) = (n_1 - n_2 + 1) (J_+ \left| n_1, n_2 \right\rangle)$$

Furthermore

$$J_+ \left| n_1, n_2 \right\rangle = a_1^\dagger a_2 \left| n_1, n_2 \right\rangle = \frac{1}{\sqrt{n_1! n_2!}} \left(a_1^\dagger \right)^{n_1 + 1} a_2 \left(a_2^\dagger \right)^{n_2} \left| 0 \right\rangle$$

Use

$$[a_2, \left(a_2^\dagger \right)^{n_2}] = n_2 \left(a_2^\dagger \right)^{n_2 - 1}$$

we get

$$J_+ \left| n_1, n_2 \right\rangle = \frac{1}{\sqrt{n_1! n_2!}} \left(a_1^\dagger \right)^{n_1 + 1} n_2 \left(a_2^\dagger \right)^{n_2 - 1} \left| 0 \right\rangle = \sqrt{(n_1 + 1)(n_2)} \left| n_1 + 1, n_2 - 1 \right\rangle$$

From

$$J = \frac{1}{2} (n_1 + n_2), \qquad m = \frac{1}{2} (n_1 - n_2)$$

we see that

$$n_1 = J + m, \qquad n_2 = J - m$$

and

$$J_+ \left| n_1, n_2 \right\rangle = \sqrt{(J + m + 1)(J - m)} \left| n_1 + 1, n_2 - 1 \right\rangle$$

Quantum Field Theory
Homework 3 solution

1. The Dirac equation for free particle is given by,

$$(i\gamma^{\mu}\partial_{\mu} - m)\,\psi(x) = 0$$

Under the parity transformation the space-time coordiante transform as

$$x^{\mu} \to x'^{\mu} = (x_0, -x_1, -x_2, -x_3)$$

The Dirac equation in the new coordinate system is of the form,

$$(i\gamma^{\mu}\partial'_{\mu} - m)\,\psi'(x') = 0$$

Find the relation between $\psi(x)$ and $\psi'(x')$.

Solution: For the parity transformation, the Lorentz transformation is of the form,

$$\Lambda_{\mu}^{\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Then from

$$S^{-1}\gamma_{\mu}S = \Lambda_{\mu}^{\nu}\gamma_{\nu}$$

we see that

$$S^{-1}\gamma_0 S = \gamma_0, \qquad S^{-1}\gamma_i S = -\gamma_i, \qquad i = 1, 2, 3$$

Clearly,

$$S = \gamma_0, \qquad \text{and} \qquad \psi'(x') = \gamma_0\psi(x)$$

It is easy to see that

$$\bar{\psi}'\psi' = \bar{\psi}\psi, \qquad \bar{\psi}'\gamma_5\psi' = -\bar{\psi}\gamma_5\psi, \qquad \bar{\psi}'\gamma_{\mu}\psi' = \bar{\psi}\gamma^{\mu}\psi, \qquad \bar{\psi}'\gamma_{\mu}\gamma_5\psi' = -\bar{\psi}\gamma^{\mu}\gamma_5\psi, \qquad \bar{\psi}'\sigma_{\mu\nu}\psi' = \bar{\psi}\sigma^{\mu\nu}\psi,$$

2. The left-handed and right-handed components of a Dirac particle are defined by,

$$\psi_L \equiv \frac{1}{2}(1-\gamma_5)\,\psi, \qquad \psi_R \equiv \frac{1}{2}(1+\gamma_5)\,\psi$$

where γ_5 is defined by

$$\gamma_5 \equiv \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

(a) Show that

$$\{\gamma_5, \gamma_{\mu}\} = 0, \qquad \text{and} \quad \gamma_5^2 = 1$$

(b) Show that ψ_L, ψ_R are eigenstates of γ_5 matrix. What are the eigenvalues?

(c) Are they eigenstates of parity operator?

(d) Write the u spinor in the form,

$$u(p,s) = N\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_s$$

where N is some normalization constant and χ_s is an arbitrary 2 component spinor. Show that if we choose χ_s to be eigenstate of $\vec{\sigma}\cdot\vec{p}$,

$$\left(\vec{\sigma}\cdot\hat{p}\right)\chi_s = \frac{1}{2}\chi_s$$

then $u(p,s)$ is an eigenstate of the helicity operator $\lambda = \vec{S}\cdot\hat{p}$ where $\vec{S}$ is the spin operator given by

$$\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

Solution:
a)

$$\{\gamma_5, \gamma^0\} = i\{\gamma^0\gamma^1\gamma^2\gamma^3, \gamma^0\} = i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 + \gamma^0\gamma^0\gamma^1\gamma^2\gamma^3) = i(-\gamma^1\gamma^2\gamma^3 + \gamma^1\gamma^2\gamma^3) = 0$$

$$\begin{aligned}\gamma_5^2 &= i^2\left(\gamma^0\gamma^1\gamma^2\gamma^3\right)\left(\gamma^0\gamma^1\gamma^2\gamma^3\right) = -(-)\left(\gamma^1\gamma^2\gamma^3\right)\left(\gamma^1\gamma^2\gamma^3\right) = (-)\left(\gamma^2\gamma^3\right)\gamma^2\gamma^3 \\ &= (-)\left(\gamma^2\gamma^3\right)\gamma^2\gamma^3 = -\left(\gamma^3\right)^2 = 1\end{aligned}$$

b)

$$\gamma_5\psi_L \equiv \gamma_5\frac{1}{2}(1-\gamma_5)\psi = \frac{1}{2}(\gamma_5 - 1)\psi = -\psi_L$$

Similarly

$$\gamma_5\psi_R = \psi_R$$

c) Under the parity we have

$$P\psi = \gamma_0\psi$$

Then

$$P\psi_L = \gamma_0\psi_L = \psi_R, \qquad P\psi_R = \gamma_0\psi_R = \psi_L$$

d) In the standard representation

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Thus

$$\begin{aligned}u_L(p) &= \frac{1}{2}(1-\gamma_5)\,u(p,-) = N\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_- \\ &= N\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{-p}{E} \end{pmatrix}\chi_- = N\frac{1}{2}\left(\frac{E+p}{E}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}\chi_- = N\begin{pmatrix} 1 \\ -1 \end{pmatrix}\chi_-\end{aligned}$$

3. Consider a one-dimensional string with length L which satisfies the wave equaiton,

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2}$$

(a) Find the solutions of this wave equation with the boundary conditions,

$$\phi(0,t) = \phi(L,t) = 0$$

(b) Find the Lagrangian density which will give this wave equation as the equation of motion.

(c) From the Lagrangian density find the conjugate momenta and impose the quantization conditions. Also find the Hamiltonian.

(d) Find the eigenvalues of the Hamiltonian.

Solution :
a)Write $\phi(x,t) = \psi(x)\,e^{-iEt}$.Then

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{E^2}{v^2}$$

Plane wave solution $\phi = e^{ikx}$, gives

$$E^2 = v^2k^2$$

Or

$$E = \pm\omega, \qquad \omega = kv$$

For the boundary condition $\psi(0) = \psi(L) = 0$,we take

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$$

so that

$$\int_0^L \psi_n(x)\,\psi_m(x)\,dx = \delta_{nm}$$

Note that the energy eigenvalues are

$$E_n = \pm\omega_n, \qquad \omega_n = \frac{n\pi v}{L}$$

b)The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 - \frac{v^2}{2}(\partial_x\phi)^2$$

Euler-Lagrange Eq

$$\partial_x \frac{\partial\mathcal{L}}{\partial(\partial_x\phi)} - \partial_t \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = 0, \qquad \Longrightarrow \qquad \frac{\partial^2\phi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2}$$

c)Conjugate mometua

$$\pi(x,t) = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = (\partial_t\phi)$$

Hamiltonian

$$H = \pi\phi - \mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 + \frac{v^2}{2}(\partial_x\phi)^2$$

Commutation relations

$$[\phi(x,t)\,,\ \partial_t\phi(y,t)] = i\delta(x-y)$$

d) Mode expansion

$$\phi(x,t) = \sum_n \sqrt{\frac{2}{L}}\left[a_n \sin\frac{n\pi x}{L}e^{-i\omega_n t} + a_n^\dagger \sin\frac{n\pi x}{L}e^{i\omega_n t}\right]\frac{1}{\sqrt{2\omega_n}}$$

$$\partial_t\phi(x,t) = \sum_n \sqrt{\frac{2}{L}}(-i\omega_n)\left[a_n \sin\frac{n\pi x}{L}e^{-i\omega_n t} - a_n^\dagger \sin\frac{n\pi x}{L}e^{i\omega_n t}\right]\frac{1}{\sqrt{2\omega_n}}$$

$$a_n = \frac{i}{\sqrt{2\omega_n}}\int_0^L dx\,[(-i\omega_n)\phi(x,t) + \partial_t\phi(x,t)]\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}e^{i\omega_n t}$$

Then

$$\begin{aligned}[a_n, a_m^\dagger] &= \frac{1}{\sqrt{2\omega_n 2\omega_m}}\int_0^L dxdy\left(\frac{2}{L}\right)\sin\frac{n\pi x}{L}e^{-i\omega_n t}\sin\frac{n\pi y}{L}e^{i\omega_m t} \\ &\quad [(-i\omega_n)\phi(x,t) + \partial_t\phi(x,t)\,,(i\omega_m)\phi(y,t) + \partial_t\phi(y,t)]\end{aligned}$$

Or

$$\begin{aligned}[a_n, a_m^\dagger] &= \frac{1}{\sqrt{2\omega_n 2\omega_m}}\int_0^L dx\left(\frac{2}{L}\right)\sin\frac{n\pi x}{L}e^{-i\omega_n t}\sin\frac{m\pi x}{L}e^{i\omega_m t}(\omega_n + \omega_m) \\ &= \delta_{nm}\end{aligned}$$

Hamiltonian is

$$H = \int_0^L dx\left[\frac{1}{2}(\partial_t\phi)^2 + \frac{v^2}{2}(\partial_x\phi)^2\right]$$

The first term is

$$\begin{aligned}\int_0^L dx\frac{1}{2}(\partial_t\phi)^2 &= \sum_{n,m}\frac{2}{L}(-\omega_n)(\omega_m)\frac{1}{\sqrt{2\omega_n}}\frac{1}{\sqrt{2\omega_m}}\int_0^L dx\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\left[a_n e^{-i\omega_n t} - a_n^\dagger e^{i\omega_n t}\right]\times\left[a_m e^{-i\omega_m t} - \right. \\ &= \frac{1}{2}\sum_n(-\omega_n)\frac{1}{2}\left[a_n a_n e^{-2i\omega_n t} + a_n^\dagger a_n^\dagger e^{2i\omega_n t} - a_n a_n^\dagger - a_n^\dagger a_n\right]\end{aligned}$$

and the second term is

$$\begin{aligned}v^2\int_0^L dx\frac{1}{2}(\partial_x\phi)^2 &= \sum_{n,m}\frac{2}{L}\left(\frac{n\pi}{L}\right)\left(\frac{m\pi}{L}\right)\frac{1}{\sqrt{2\omega_n}}\frac{1}{\sqrt{2\omega_m}}\int_0^L dx\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\left[a_n e^{-i\omega_n t} + a_n^\dagger e^{i\omega_n t}\right]\times\left[a_m e^{-i\omega} \right. \\ &= \frac{1}{2}v^2\sum_n\left(\frac{n\pi}{L}\right)^2\frac{1}{2\omega_n}\left[a_n a_n e^{-2i\omega_n t} + a_n^\dagger a_n^\dagger e^{2i\omega_n t} + a_n a_n^\dagger + a_n^\dagger a_n\right]\end{aligned}$$

The Hamiltonian is then

$$H=\int_0^L dx\left[\frac{1}{2}\left(\partial_t\phi\right)^2+\frac{v^2}{2}\left(\partial_x\phi\right)^2\right]=\frac{1}{2}\sum_n\omega_n\left(a_na_n^{\dagger}+a_n^{\dagger}a_n\right)$$

4. Consider the Lagrangian density given by

$$\mathcal{L}=\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}\mu^2\phi^2+J\left(x\right)\phi,\qquad J\left(x\right)\quad \text{arbitray function}$$

(a) Show that the equation of motion is of the form,

$$\left(\partial^\mu\partial_\mu+\mu^2\right)\phi\left(x\right)=J\left(x\right)$$

(b) Find the conjugate momenta and impose the quantization conditions.

(c) Find the creation and annihilation operators.

Solution:
a) Equation of motion

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}=\frac{\partial\mathcal{L}}{\partial\phi},\qquad\Longrightarrow\qquad\partial^\mu\partial_\mu\phi+\mu^2\phi=J\phi,$$

b)
Conjugate momentum

$$\pi=\frac{\partial\mathcal{L}}{\partial\left(\partial_0\phi\right)}=\partial_0\phi$$

Quantization

$$\left[\phi\left(x,t\right),\ \pi\left(y,t\right)\right]=i\delta^3\left(x-y\right)$$

Define

$$\left(\partial^\mu\partial_\mu+\mu^2\right)\Delta\left(x-y\right)=\delta^4\left(x-y\right)$$

Then

$$\phi\left(x\right)=\phi_0\left(x\right)+\int\Delta\left(x-y\right)J\left(y\right)d^4y=\phi_0\left(x\right)+\phi_{cl}\left(x\right)$$

where $\phi_0\left(x\right)$ satisfies the homogeneous equaiton

$$\left(\partial^\mu\partial_\mu+\mu^2\right)\phi_0\left(x\right)=0$$

and

$$\phi_{cl}\left(x\right)=\int\Delta\left(x-y\right)J\left(y\right)d^4y$$

So $\phi_0\left(x\right)$ can be expanded in terms of plane waves

$$\phi_0\left(x\right)=\int\frac{d^3p}{\sqrt{\left(2\pi\right)^3 2\omega_p}}\left[a\left(p\right)e^{-ipx}+a^{\dagger}\left(p\right)e^{ipx}\right]$$

Then we can solve for $a\left(p\right)$ to write

$$\begin{aligned}a\left(p\right)&=\int\frac{d^3x}{\sqrt{\left(2\pi\right)^3 2\omega_p}}\left\{e^{ipx}\overleftrightarrow{\partial_0}\phi_0\left(x\right)\right\}\\&=\int\frac{d^3x}{\sqrt{\left(2\pi\right)^3 2\omega_p}}\left\{e^{ipx}\overleftrightarrow{\partial_0}\left(\phi\left(x\right)-\int\Delta\left(x-y\right)J\left(y\right)d^4y\right)\right\}\end{aligned}$$

Note that the last term is a c-number and will not effect the commutation relation.
We can write the action as

$$\begin{aligned}S&=\int d^4x\mathcal{L}=\int d^4x\left[-\frac{1}{2}\phi\left(\partial^2+\mu^2\right)\phi+J\phi\right]\\&=\int d^4x\left[-\frac{1}{2}\left(\phi_0+\phi_{cl}\right)\left(\partial^2+\mu^2\right)\left(\phi_0+\phi_{cl}\right)+J\left(\phi_0+\phi_{cl}\right)\right]\\&=\int d^4x\left[-\frac{1}{2}\phi_0\left(\partial^2+\mu^2\right)\phi_0-\phi_0\left(\left(\partial^2+\mu^2\right)\phi_{cl}-J\right)-\frac{1}{2}\phi_{cl}\left(\partial^2+\mu^2\right)\phi_{cl}+J\phi_{cl}\right]\\&=\int d^4x\left[-\frac{1}{2}\phi_0\left(\partial^2+\mu^2\right)\phi_0-\frac{1}{2}\phi_{cl}\left(\partial^2+\mu^2\right)\phi_{cl}+J\phi_{cl}\right]\end{aligned}$$

Note that

$$\left(\partial^2 + \mu^2\right) \phi_{cl} = J$$

We get

$$S = \int d^4x \left[-\frac{1}{2}\phi_0 \left(\partial^2 + \mu^2\right) \phi_0 + J\phi_{cl} \right]$$

5. Let ϕ be a free scalar field satisfying the field equation,

$$\left(\partial^\mu \partial_\mu + \mu^2\right) \phi(x) = 0$$

(a) Show that the propagator defined by

$$\Delta_F(x-y) \equiv \langle 0 | T(\phi(x)\phi(y)) | 0 \rangle = \theta(x_0 - y_0)\phi(x)\phi(y) + \theta(y_0 - x_0)\phi(y)\phi(x)$$

can be written as

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \frac{i}{k^2 - \mu^2 + i\varepsilon}$$

(b) Show that the unequal time commutator for these free fields is given by

$$i\Delta(x-y) \equiv [\phi(x), \phi(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)} \right]$$

(c) Show that $\Delta(x-y) = 0$ for space-like separation, i.e.

$$\Delta(x-y) = 0, \qquad \text{if} \quad (x-y)^2 < 0$$

Solution:
a)

$$\begin{aligned} i\Delta(x,y) &= \langle 0 | T(\phi(x)\phi(y)) | 0 \rangle \\ &= \theta(x_0 - y_0) \langle 0 | \phi(x)\phi(y) | 0 \rangle + \theta(y_0 - x_0) \langle 0 | \phi(y)\phi(x) | 0 \rangle \end{aligned}$$

Using the mode expansion, we see that

$$\begin{aligned} \langle 0 | \phi(x)\phi(y) | 0 \rangle &= \int \frac{d^3k d^3k'}{(2\pi)^3 \sqrt{2\omega_k 2\omega_{k'}}} \left\langle 0 \left| [a(k)e^{-ikx}] a^+(k') e^{ik'y} \right| 0 \right\rangle \\ &= \int \frac{d^3k d^3k'}{(2\pi)^3 2\omega_k} \delta^3(k-k') e^{-ikx + ik'y} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik(x-y)} \end{aligned}$$

$$i\Delta(x,y) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[\theta(x_0 - y_0) e^{-ik(x-y)} + \theta(y_0 - x_0) e^{ik(x-y)} \right]$$

Note that

$$\frac{1}{2\pi} \int \frac{dk_0}{k_0^2 - \omega^2 + i\varepsilon} e^{-ik_0(x_0 - y_x{}')} = \begin{cases} -i\frac{1}{2\omega} e^{-i\omega(x_0 - y_0)} & \text{for} \quad x_0 > y_0 \\ -i\frac{1}{2\omega} e^{i\omega(x_0 - y_0)} & \text{for} \quad x_0 < y_0 \end{cases}$$

We then get

$$\begin{aligned} \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x'-y)}}{k^2 + i\varepsilon} &= -i \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[\theta(t-t') e^{-ik(x-x')} + \theta(t-t') e^{ik(x-x')} \right] \\ &= i\Delta(x,y) \end{aligned}$$

b) From part a) we see immediately that

$$i\Delta(x-y) \equiv [\phi(x), \phi(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)} \right]$$

c) For space like separation $(x-y)^2 < 0$, we can chose a frame such that $x-y$ has only spatial component $x - y = \left(0, \vec{x} - \vec{y}\right)$.Then

$$[\phi(x), \phi(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[e^{i\vec{k}\cdot\left(\vec{x}-\vec{y}\right)} - e^{-i\vec{k}\cdot\left(\vec{x}-\vec{y}\right)} \right] = 0$$

where we have change the integration variable $\vec{k}$ to $-\vec{k}$ in the second term.

Quantum Field Theory

Ling-Fong Li

December 5, 2014

Homework set 4, Solution

1. Dirac equation for electron moving in the electromagnetic field can be obtained from the free Dirac equation by the replacement $i\partial_\mu \longrightarrow i\partial_\mu - eA_\mu$,

$$\left[\gamma^\mu \left(i\partial_\mu - eA_\mu\right) - m\right]\psi\left(\vec{x}, t\right) = 0$$

Then the equation for the positron is

$$\left[\gamma^\mu \left(i\partial_\mu + eA_\mu\right) - m\right]\psi_c\left(\vec{x}, t\right) = 0$$

Assume that ψ_c is related to ψ by

$$\psi_c = \tilde{C}\psi^*$$

$\tilde{C}$ is called the charge conjugation matrix.

(a) Find $\tilde{C}$ in terms of Dirac γ matrices.

(b) For the $v-$spinor of the form,

$$v\left(p, s\right) = N\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\chi_s$$

Compute its charge conjugate $v_c\left(p, s\right) = \tilde{C}v^*\left(p, s\right)$

(c) To implement the charge conjugation for the fermion field, we write

$$\psi_c = C\psi C^{-1} = \tilde{C}\psi^*$$

where C is the charge conjugation operator. Find the relation between $\bar{\psi}_c\gamma^\mu\psi_c$ and $\bar{\psi}\gamma^\mu\psi$.

Solution:

a) Dirac equation for a charged particle in em field is of the form,

$$\left[\gamma^\mu \left(i\partial_\mu - eA_\mu\right) - m\right]\psi = 0$$

On the other hand the equation for positron is

$$\left[\gamma^\mu \left(i\partial_\mu + eA_\mu\right) - m\right]\psi_c = 0$$

Take the complex conjugate of Dirac equation we get

$$\left[-\left(\gamma^\mu\right)^* \left(i\partial_\mu + eA_\mu\right) - m\right]\psi^* = 0$$

If we assume ψ_c is related to ψ^* by

$$\psi_c = \widetilde{C}\psi^*$$

then

$$\widetilde{C}^{-1}\gamma^\mu\widetilde{C} = -\gamma^{\mu *}$$

In the standard notation where $\gamma_0, \gamma_1, \gamma_3$ are real and γ_2 *is* imaginary, $\widetilde{C}$ can be taken as

$$\widetilde{C} = i\gamma_2 = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}$$

which has the properties,

$$\widetilde{C}^{-1} = \widetilde{C}^\dagger = \widetilde{C}$$

b) From the $v-spinor$ of the form,

$$v(p,s)=N\begin{pmatrix}\dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\\1\end{pmatrix}\chi_s,\qquad s=\pm$$

we get

$$\begin{aligned}v_c(p,s) &= \widetilde{C}v^*(p,s)=\begin{pmatrix}0 & -i\sigma_2\\ i\sigma_2 & \end{pmatrix}N\begin{pmatrix}\dfrac{\vec{\sigma}^*\cdot\vec{p}}{E+m}\\1\end{pmatrix}\chi_s^*=N\begin{pmatrix}-i\sigma_2\\ i\sigma_2\left(\dfrac{\vec{\sigma}^*\cdot\vec{p}}{E+m}\right)\end{pmatrix}\chi_s\\ &= N\begin{pmatrix}1\\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\end{pmatrix}(-i\sigma_2\chi_s)=N\begin{pmatrix}1\\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E+m}\end{pmatrix}(s)\,\chi_{-s}=(s)\,u(p,-s)\end{aligned}$$

where we have used the relations

$$\sigma_2\vec{\sigma}^*\sigma_2=-\vec{\sigma},\qquad -i\sigma_2\chi_s=(s)\,\chi_{-s}$$

Note that the spin component is flipped under charge conjugation.
In terms of creation and annihilation operators, we have

$$\psi=\sum_s\int\frac{d^3p}{\sqrt{(2\pi)^3\,2E_p}}\left[b(p,s)\,u(p,s)\,e^{-ip\cdot x}+d^\dagger(p,s)\,v(p,s)\,e^{ip\cdot x}\right]$$

and the charge conjugate field is

$$\begin{aligned}\psi_c &= \widetilde{C}\left(\psi^\dagger\right)^T=\sum_s\int\frac{d^3p}{\sqrt{(2\pi)^3\,2E_p}}\left[b^\dagger(p,s)\,\widetilde{C}u^*(p,s)\,e^{ip\cdot x}+d(p,s)\,\widetilde{C}v^*(p,s)\,e^{-ip\cdot x}\right]\\ &= \sum_s\int\frac{d^3p}{\sqrt{(2\pi)^3\,2E_p}}\left[b^\dagger(p,s)\,v(p,-s)\,e^{ip\cdot x}+d(p,s)\,u(p,-s)\,e^{-ip\cdot x}\right]\end{aligned}$$

Write

$$\psi_c=C\psi C^{-1}$$

then

$$Cb(p,s)\,C^{-1}=d(p,-s)\,,\qquad Cd^\dagger(p,s)\,C^{-1}=b^\dagger(p,-s)$$

c) From

$$\psi_c=\widetilde{C}\psi^*$$

we get

$$\psi_c^\dagger=\psi^T\widetilde{C}^\dagger,\qquad\text{and}\qquad\bar{\psi}_c=\psi^T\widetilde{C}^\dagger\gamma_0\tag{1}$$

Then

$$\begin{aligned}\bar{\psi}_c\gamma^\mu\psi_c &= \psi^T\widetilde{C}^\dagger\gamma_0\gamma^\mu\widetilde{C}\psi^*=\psi^T\gamma_0\left(-\widetilde{C}^\dagger\gamma^\mu\widetilde{C}\right)\psi^*=\psi^T\gamma_0\,(\gamma^\mu)^*\,\psi^*\\ &= \left[\psi^T\gamma_0\,(\gamma^\mu)^*\,\psi^*\right]^T=\psi^\dagger\,(\gamma^\mu)^\dagger\,\gamma_0\psi=-\bar{\psi}\gamma^\mu\psi\end{aligned}$$

where we have used the property that fermion fields anti-commute. This means the 4-vector current for the anti particles is negative of that for the particle. In other words, the 4-vector current is odd under charge conjugation.
Similarly for the scalar current we have

$$\begin{aligned}\bar{\psi}_c\psi_c &= \psi^T\widetilde{C}^\dagger\gamma_0\widetilde{C}\psi^*=\psi^T\gamma_0\left(-\widetilde{C}^\dagger\widetilde{C}\right)\psi^*=-\psi^T\gamma_0\psi^*\\ &= -\left[\psi^T\gamma_0\psi^*\right]^T=\psi^\dagger\gamma_0\psi=\bar{\psi}\psi\end{aligned}$$

and for the psudoscalar current

$$\begin{aligned}\bar{\psi}_c\gamma_5\psi_c &= \psi^T\widetilde{C}^\dagger\gamma_0\gamma_5\widetilde{C}\psi^*=\psi^T\gamma_0\gamma_5\left(\widetilde{C}^\dagger\widetilde{C}\right)\psi^*=\psi^T\gamma_0\gamma_5\psi^*\\ &= \left[\psi^T\gamma_0\gamma_5\psi^*\right]^T=-\psi^\dagger\gamma_5\gamma_0\psi=\bar{\psi}\gamma_5\psi\end{aligned}$$

This shows that the scalar current is even under charge conjugation.

$$\begin{aligned}\bar{\psi}_c\gamma^\mu\gamma_5\psi_c &= \psi^T\widetilde{C}^\dagger\gamma_0\gamma^\mu\gamma_5\widetilde{C}\psi^* = \psi^T\gamma_0\left(-\widetilde{C}^\dagger\gamma^\mu\gamma_5\widetilde{C}\right)\psi^* = -\psi^T\gamma_0\left(\gamma^\mu\right)^*\gamma_5\psi^* \\ &= -\left[\psi^T\gamma_0\left(\gamma^\mu\right)^*\gamma_5\psi^*\right]^T = \psi^\dagger\gamma_5\left(\gamma^\mu\right)^\dagger\gamma_0\psi = \bar{\psi}\gamma^\mu\gamma_5\psi\end{aligned}$$

2. Consider a free scalar field $\phi(x)$ where the 4-momentum operator is of the form,

$$P^\mu = \int d^3k\; k^\mu a^\dagger(k)\, a(k)$$

   (a) As a useful tool, show that for two operators A and B, the following identity holds

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots$$

   (b) Use this identity to show that

$$e^{iP\cdot x} a(k)\, e^{-iP\cdot x} = a(k)\, e^{-ik\cdot x}$$

   and

$$[P^\mu, \phi(x)] = i\partial^\mu\phi(x)$$

   (c) Let $|K\rangle$ be an eigenstate of P^μ, satisfying $P^\mu|K\rangle = K^\mu|K\rangle$. Show that

$$\langle K|\phi(x)\,\phi(y)|K\rangle = \langle K|\phi(x-y)\,\phi(0)|K\rangle$$

Solution :
a) consider the function $F(\lambda)$ defined as

$$F(\lambda) = e^{\lambda A} B e^{-\lambda A}$$

Then

$$\frac{dF}{d\lambda} = e^{\lambda A}[A,B]e^{-\lambda A}, \qquad \frac{d^2F}{d\lambda^2} = e^{\lambda A}[A,[A,B]]e^{-\lambda A}, \cdots$$

On the other hand, Taylor expansion of $F(\lambda)$,gives

$$F(\lambda) = F(0) + \lambda\frac{dF}{d\lambda}|_{\lambda=0} + \frac{\lambda^2}{2!}\frac{d^2F}{d\lambda^2}|_{\lambda=0} + \cdots$$

Setting $\lambda = 1$, we get

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots$$

b) From the identity in a) we get

$$e^{iP\cdot x} a(k)\, e^{-iP\cdot x} = a(k) + ix^\mu\left[P_\mu, a(k)\right] + \frac{1}{2}x^\nu x^\mu[P_\nu,[P_\mu, a(k)]] + \cdots$$

Now we calculate the commutators,

$$[P_\mu, a(k)] = \int d^3k'\; k'^\mu[a^\dagger(k')\, a(k'), a(k)] = k^\mu a(k)$$

and

$$[P_\nu,[P_\mu, a(k)]] = k^\mu k^\nu a(k)$$

Then

$$e^{iP\cdot x} a(k)\, e^{-iP\cdot x} = a(k)\left[1 + ik\cdot x + \frac{1}{2}\left(ik\cdot x\right)^2 + \cdots\right] = e^{ik\cdot x} a(k)$$

Take Hermitian conjugate we get

$$e^{iP\cdot x} a^\dagger(k)\, e^{-iP\cdot x} = e^{-ik\cdot x} a^\dagger(k)$$

From the formula for P_μ and

$$\phi(x) = \int \frac{d^3p}{\sqrt{2\omega_p (2\pi)^3}} \left[a(p) e^{-ipx} + a^\dagger(p) e^{ipx}\right]$$

we get

$$\begin{aligned}
[P^\mu, \phi(x)] &= \int \frac{d^3p}{\sqrt{2\omega_p (2\pi)^3}} \int d^3k' \, k'^\mu [a^\dagger(k') a(k'), a(p) e^{-ipx} + a^\dagger(p) e^{ipx}] \\
&= \int \frac{d^3p}{\sqrt{2\omega_p (2\pi)^3}} \int d^3k' \, k'^\mu [a(k') e^{-ipx} - a^\dagger(p) e^{ipx}] \delta^3(p-k') \\
&= \int \frac{d^3p}{\sqrt{2\omega_p (2\pi)^3}} p^\mu [a(p) e^{-ipx} - a^\dagger(p) e^{ipx}] = i\partial^\mu \phi
\end{aligned}$$

From this we can show that

$$e^{iP\cdot a} \phi(x) e^{-iP\cdot a} = \phi(x) + ia^\mu [P_\mu, \phi(x)] + \cdots = \phi(x-a)$$

c) Write

$$\phi(y) = e^{-iP\cdot y} \phi(0) e^{iP\cdot y}$$

Then

$$\begin{aligned}
\langle K | \phi(x) \phi(y) | K \rangle &= \left\langle K \left| \phi(x) e^{-iP\cdot y} \phi(0) e^{iP\cdot y} \right| K \right\rangle = \left\langle K \left| e^{-iP\cdot y} e^{iP\cdot y} \phi(x) e^{-iP\cdot y} \phi(0) \right| K \right\rangle e^{iK\cdot y} \\
&= e^{-iK\cdot y} \langle K | \phi(x-y) \phi(0) | K \rangle e^{iK\cdot y} = \langle K | \phi(x-y) \phi(0) | K \rangle
\end{aligned}$$

3. The propagator for a massless scalar field can be written in the form,

$$\Delta_F(x) = \int \frac{d^4x}{(2\pi)^4} \frac{e^{ik\cdot x}}{k^2 + i\varepsilon}$$

Carrying out the integration to show that

$$\Delta_F(x) = \frac{i}{4\pi^2} \frac{1}{x^2 - i\varepsilon}$$

Solution:

$$\Delta_F(x) = \int \frac{d^4x}{(2\pi)^4} \frac{e^{ik\cdot x}}{k^2 + i\varepsilon} = \int \frac{d^3k}{(2\pi)^4} e^{i\vec{k}\cdot\vec{x}} \int \frac{dk_0 e^{-ik_0 x_0}}{k_0^2 - \vec{k}^2 + i\varepsilon}$$

The k_0 integration can be performed by the standard contour method,

$$\int \frac{dk_0 e^{-ik_0 x_0}}{k_0^2 - \vec{k}^2 + i\varepsilon} = \frac{-i\pi}{\left|\vec{k}\right|} [\theta(x_0) e^{-i\left|\vec{k}\right| x_0} + \theta(-x_0) e^{i\left|\vec{k}\right| x_0}]$$

We then have

$$\Delta_F(x) = -\frac{1}{8\pi^2 r} \int_0^\infty dk \left(e^{ikr} - e^{-ikr}\right) [\theta(x_0) e^{-ikx_0} + \theta(-x_0) e^{ikx_0}]$$

Using the identity

$$\int_0^\infty e^{\pm i\alpha\tau} d\tau = \int_0^\infty e^{\pm i(\alpha \pm i\varepsilon)\tau} d\tau = \frac{\mp 1}{i(\alpha \pm i\varepsilon)}$$

we get

$$\begin{aligned}
\Delta_F(x) &= -\frac{1}{8\pi^2 r} \left[\theta(x_0) \left(\frac{1}{r - x_0 + i\varepsilon} + \frac{1}{r + x_0 - i\varepsilon}\right) + \theta(-x_0) \left(\frac{1}{r + x_0 + i\varepsilon} + \frac{1}{r - x_0 - i\varepsilon}\right)\right] \\
&= \frac{-i}{4\pi^2} \left[\frac{\theta(x_0)}{r^2 - x_0^2 + i\varepsilon x_0} + \frac{\theta(-x_0)}{r^2 - x_0^2 - i\varepsilon x_0}\right] = \frac{i}{4\pi^2} \frac{1}{(x^2 - i\varepsilon)}
\end{aligned}$$

4. In thequantization of free electromagnetic fields the mode expansion is of the form,

$$\overrightarrow{A}(\vec{x},t)=\int\frac{d^3k}{\sqrt{2\omega(2\pi)^3}}\sum_{\lambda}\overrightarrow{\epsilon}(\vec{k},\lambda)[a(k,\lambda)e^{-ikx}+a^{\dagger}(k,\lambda)e^{ikx}]\qquad w=k_0=|\overrightarrow{k}|$$

where

$$\vec{\epsilon}(k,\lambda),\lambda=1,2\quad\text{with }\vec{k}\cdot\vec{\epsilon}(k,\lambda)=0$$

The quantization condition is of the form,

$$[\partial_0 A_i(\overrightarrow{x},t),\ A_j(\overrightarrow{x}',t)]=-i\delta_{ij}^{tr}\delta_{ij}^{tr}(x-x')$$

Solve for $a(k,\lambda)$ and $a^{+}(k,\lambda)$ and compute the commutator,

$$[a(k,\lambda),\ a^{\dagger}(k',\lambda')]$$

Solution:

$$a(k,\lambda)=i\int\frac{d^3x}{\sqrt{(2\pi)^3 2\omega}}[e^{ik\cdot x}\overleftrightarrow{\partial_0}\vec{\epsilon}(k,\lambda)\cdot\vec{A}(x)]$$

$$a^{\dagger}(k,\lambda)=-i\int\frac{d^3x}{\sqrt{(2\pi)^3 2\omega}}[e^{-ik\cdot x}\overleftrightarrow{\partial_0}\vec{\epsilon}(k,\lambda)\cdot\vec{A}(x)]$$

Commutation relations,

$$\begin{aligned}[a(k,\lambda),\ a^{\dagger}(k',\lambda')] &= \int\frac{d^3xd^3x'e^{ikx}e^{-ik'x'}}{\sqrt{(2\pi)^3\,2w_k\,(2\pi)^3\,2w_{k'}}}\Big[\vec{\epsilon}(k,\lambda)\cdot\partial_0\vec{A}(x)-ik_0\vec{\epsilon}(k,\lambda)\cdot\vec{A}(x),\ \vec{\epsilon}(k',\lambda')\cdot\partial_0\vec{A}(x')+ik_0'\vec{\epsilon}(k',\lambda')\\
&= \int\frac{d^3xd^3x'e^{ikx}e^{-ik'x'}}{\sqrt{(2\pi)^3\,2w_k\,(2\pi)^3\,2w_{k'}}}\{\varepsilon_i(k,\lambda)ik_0'\varepsilon_j(k',\lambda')[\partial_0 A_i(x),\ A_j(x')]-ik_0\varepsilon_i(k,\lambda)\varepsilon_j(k',\lambda')[A_i(x),\ \partial\\
&= \int\frac{d^3xd^3x'e^{ikx}e^{-ik'x'}}{\sqrt{(2\pi)^3\,2w_k\,(2\pi)^3\,2w_{k'}}}\{\varepsilon_i(k,\lambda)\varepsilon_j(k',\lambda')\,(-i)\,\delta_{ij}^{tr}(x-x')\,(ik_0'+ik_0)\}\end{aligned}$$

Note that

$$\begin{aligned}l &= \int d^3xd^3x'e^{i\overrightarrow{k}\cdot\overrightarrow{x}}e^{-i\overrightarrow{k'}\cdot\overrightarrow{x}'}\varepsilon_i(k,\lambda)\varepsilon_j(k',\lambda')\delta_{ij}^{tr}(x-x')=\int d^3xd^3x'e^{i\overrightarrow{k}\cdot\overrightarrow{x}}e^{-i\overrightarrow{k'}\cdot\overrightarrow{x}'}\varepsilon_i(k,\lambda)\varepsilon_j(k',\lambda')\int d^3q e^{i\overrightarrow{q}\cdot(\overrightarrow{x}-\overrightarrow{x}')}(\delta\\
&= \int d^3xd^3x'e^{i(\overrightarrow{k}-\overrightarrow{q})\cdot\overrightarrow{x}}e^{-i(\overrightarrow{k'}-\overrightarrow{q})\cdot\overrightarrow{x}'}\varepsilon_i(k,\lambda)\varepsilon_j(k',\lambda')\int d^3q(\delta_{ij}-\frac{q_iq_j}{q^2})=\int d^3q\delta^3(\overrightarrow{k}-\overrightarrow{q})\delta(\overrightarrow{k'}-\overrightarrow{q})\varepsilon_i(k,\lambda)\varepsilon_j(k',\lambda')(\\
&= \delta^3(\vec{k}-\vec{k}')(\delta_{ij}-\frac{k_ik_j}{k^2})\varepsilon_i(k,\lambda)\varepsilon_j(k',\lambda')=\delta^3(\vec{k}-\vec{k}')[\vec{\epsilon}(k,\lambda)\cdot\vec{\epsilon}(k,\lambda')-\frac{1}{k^2}\vec{\epsilon}(k,\lambda)\cdot\overrightarrow{k}\left(\vec{\epsilon}(k,\lambda')\cdot\overrightarrow{k}\right)]\\
&= \delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}')\end{aligned}$$

where we have used

$$\vec{\epsilon}(k,\lambda)\cdot\vec{\epsilon}(k,\lambda')=\delta_{\lambda\lambda'},\qquad\vec{\epsilon}(k,\lambda)\cdot\overrightarrow{k}=0$$

Quantum Field Theory

Ling-Fong Li

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Homework set 5, Solution

1. Consider the reaction

$$e^{+}(p')+e^{-}(p)\to\mu^{+}(k')+\mu^{-}(k)$$

(a) The spin averaged probability is of the form

$$\frac{1}{4}\sum_{spin}\left|M(e^{+}e^{-}\to\mu^{+}\mu^{-})\right|^{2}=\frac{e^{4}}{q^{4}}Tr\left[\left(\not p'-m_{e}\right)\gamma^{\mu}\left(\not p+m_{e}\right)\gamma^{\nu}\right]Tr\left[\left(\not k'+m_{\mu}\right)\gamma_{\mu}\left(\not k+m_{\mu}\right)\gamma^{\nu}\right]$$

Show that for energies $\gg m_{\mu}$, this can be written as

$$\frac{1}{4}\sum_{spin'}\left|M(e^{+}e^{-}\to\mu^{+}\mu^{-})\right|^{2}=8\frac{e^{4}}{q^{4}}\left[(p\cdot k)\left(p'\cdot k'\right)+(p'\cdot k)\left(p\cdot k'\right)\right]$$

(b) The phase space for this reaction is given by

$$\rho=\int(2\pi)^{4}\delta^{4}(p+p'-k-k')\frac{d^{3}k}{(2\pi)^{3}2\omega}\frac{d^{3}k'}{(2\pi)^{3}2\omega'}$$

Show that

$$\rho=\frac{d\Omega}{32\pi^{2}}$$

in the center of mass frame.

Solution:
a)

$$\begin{aligned}Tr\left[\left(\not p'-m_{e}\right)\gamma^{\mu}\left(\not p+m_{e}\right)\gamma^{\nu}\right]&=Tr\left[\not p'\gamma^{\mu}\not p\gamma^{\nu}\right]-m^{2}Tr\left[\gamma^{\mu}\gamma^{\nu}\right]\\&=4\left[p'^{\mu}p^{\nu}-g^{\mu\nu}\left(p\cdot p'\right)+p^{\mu}p'^{\nu}\right]-4m_{e}^{2}g^{\mu\nu}\\Tr\left[\left(\not k'+m_{\mu}\right)\gamma_{\mu}\left(\not k-m_{\mu}\right)\gamma^{\nu}\right]&=Tr\left[\not k'\gamma_{\mu}\not k\gamma^{\nu}\right]-m_{\mu}^{2}Tr\left[\gamma_{\mu}\gamma^{\nu}\right]\\&=4\left[k'^{\mu}k^{\nu}-g^{\mu\nu}\left(k\cdot k'\right)+k^{\mu}k'^{\nu}\right]-4m_{\mu}^{2}g^{\mu\nu}\end{aligned}$$

$$\begin{aligned}\frac{1}{4}\sum_{spin}\left|M(e^{+}e^{-}\to\mu^{+}\mu^{-})\right|^{2}&=\frac{e^{4}}{q^{4}}Tr\left[\left(\not p'-m_{e}\right)\gamma^{\mu}\left(\not p+m_{e}\right)\gamma^{\nu}\right]Tr\left[\left(\not k'+m_{\mu}\right)\gamma_{\mu}\left(\not k+m_{\mu}\right)\gamma^{\nu}\right]\\&=8\frac{e^{4}}{q^{4}}\left[(p\cdot k)\left(p'\cdot k'\right)+(p'\cdot k)\left(p\cdot k'\right)\right]\end{aligned}$$

where m_{e},m_{μ} have been neglected.
b) In the center of mass frame, we write the momenta as

$$p_{\mu}=(E,0,0,E)\,,\qquad p'_{\mu}=(E,0,0,-E)$$

$$k_{\mu}=\left(E,\vec{k}\right),\qquad k'_{\mu}=\left(E,-\vec{k}\right),\qquad\text{with }\vec{k}\cdot\hat{z}=\left|\vec{k}\right|\cos\theta$$

Then the phase space is

$$\begin{aligned}\rho&=\int(2\pi)^{4}\delta^{4}(p+p'-k-k')\frac{d^{3}k}{(2\pi)^{3}2\omega}\frac{d^{3}k'}{(2\pi)^{3}2\omega'}\\&=\frac{1}{4\pi^{2}}\int\delta\left(2E-\omega-\omega'\right)\frac{d^{3}k}{4\omega\omega'}=\frac{1}{32\pi^{2}}\int\delta\left(E-\omega\right)\frac{k^{2}dkd\Omega}{\omega^{2}}=\frac{d\Omega}{32\pi^{2}}\end{aligned}$$

2. The Lagrangian for the free photon is of the form,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Suppose we add a mass term to this Lagrangian

$$\mathcal{L}' = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mu^2 A_\mu A^\mu$$

(a) Find the equation of motion.

(b) From the equation of motion show that

$$\partial^\mu A_\mu = 0$$

and use the equation of motion to express A_0 in terms of other field variables

(c) Carry out the quantization procedure and find the eigenvalues of the Hamiltonian.

Solution :

a)

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} = -F^{\mu\nu}, \qquad \frac{\partial \mathcal{L}}{\partial A_\nu} = \mu^2 A^\nu$$

Equation of motion

$$\partial_\mu F^{\mu\nu} + \mu^2 A^\nu = 0$$

b) From equation of motion

$$\partial_\nu \partial_\mu F^{\mu\nu} + \mu^2 \partial_\nu A^\nu = 0, \qquad \Longrightarrow \qquad \partial_\nu A^\nu = 0$$

and

$$A^0 = -\frac{1}{\mu^2}\partial_i F^{i0}$$

c) Conjugate momenta

$$\pi^i = \frac{\partial \mathcal{L}}{\partial (\partial_0 A_i)} = -F^{0i}, \qquad \text{and } \ A^0 \text{ does not have conjugate momenta}$$

Commutation relation,

$$\left[\pi^i(x,t),\ A_j(x',t)\right] = -i\delta_{ij}\delta^3(x-x'), \qquad \left[A^i(x,t),\ A_j(x',t)\right] = 0, \qquad \left[\pi^i(x,t),\ \pi_j(x',t)\right] = 0$$

Note that we can write

$$A^0 = -\frac{1}{\mu^2}\partial_i F^{i0} = \frac{1}{\mu^2}\partial_i \pi^i$$

and

$$\left[A^0(x,t),\ A_j(x',t)\right] = -\frac{1}{\mu^2}\left[\partial_i \pi^i(x,t),\ A_j(x',t)\right] = -i\frac{1}{\mu^2}\partial_j \delta^3(x-x')$$

Also

$$\left[A^0(x,t),\ \pi_j(x',t)\right] = -\frac{1}{\mu^2}\left[\partial_i \pi^i(x,t),\ \pi_j(x',t)\right] = 0$$

From $\pi^i = -F^{0i} = -(\partial^0 A^i - \partial^i A^0)$, we get $\partial^0 A^i = \partial^i A^0 - \pi^i = \frac{1}{\mu^2}\partial_j \partial_i \pi^j - \pi^i$, and

$$\left[A^0(x,t),\ \partial^0 A^i(x',t)\right] = \left[A^0(x,t),\ \frac{1}{\mu^2}\partial_j \partial_i \pi^j(x',t) - \pi^i(x',t)\right] = 0$$

Equation of motion

$$\partial_\mu F^{\mu i} + \mu^2 A^i = 0, \qquad \text{or} \quad (\partial_\mu \partial^\mu + \mu^2) A^i = 0$$

Solutions are

$$\vec{A}(\vec{x},t) = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_{\lambda=1,2,3} \vec{\epsilon}(\vec{k},\lambda)[a(k,\lambda)e^{-ikx} + a^\dagger(k,\lambda)e^{ikx}], \quad \omega = \ k_0 = \sqrt{|\vec{k}|^2 + \mu^2}$$

Suppose the wave vector is in the $z-$direction,

$$k = (k_0, 0, 0, k)$$

We can choose the polarization vectors as

$$\epsilon^\mu(\vec{k},1) = (0,1,0,0), \qquad \epsilon^\mu(\vec{k},2) = (0,0,1,0), \qquad \epsilon^\mu(\vec{k},3) = \frac{1}{\mu^2}(k,1,0,k_0)$$

with

$$\epsilon(\vec{k},\lambda)\cdot\epsilon(\vec{k},\lambda') = -\delta_{\lambda\lambda'}, \qquad \lambda,\lambda' = 1,2,3$$

Since A_0 also satisfies the same Klein-Gordon equation, we can extend the expansion as

$$A^\mu(\vec{x},t) = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_{\lambda=1,2,3} \epsilon^\mu(\vec{k},\lambda)[a(k,\lambda)e^{-ikx} + a^\dagger(k,\lambda)e^{ikx}], \quad \omega = \ k_0 = \sqrt{|\overrightarrow{k}|^2+\mu^2}$$

Then

$$a(k,\lambda) = i\int \frac{d^3x}{\sqrt{(2\pi)^3 2\omega}}[e^{ik\cdot x}\overleftrightarrow{\partial_0}\varepsilon(k,\lambda)\cdot A(x)]$$

$$a^\dagger(k,\lambda) = -i\int \frac{d^3x}{\sqrt{(2\pi)^3 2\omega}}[e^{-ik\cdot x}\overleftrightarrow{\partial_0}\varepsilon(k,\lambda)\cdot A(x)]$$

Commutation relations,

$$\begin{aligned}
[a(k,\lambda),\ a^\dagger(k',\lambda')] &= \int \frac{d^3x d^3x' e^{ikx}e^{-ik'x'}}{\sqrt{(2\pi)^3\, 2w_k\, (2\pi)^3\, 2w_{k'}}}\left[\varepsilon(k,\lambda)\cdot\partial_0 A(x) - ik_0\varepsilon(k,\lambda)\cdot A(x),\ \varepsilon(k',\lambda')\cdot\partial_0 A(x') + ik_0'\varepsilon(k\right. \\
&= \int \frac{d^3x d^3x' e^{ikx}e^{-ik'x'}}{\sqrt{(2\pi)^3\, 2w_k\, (2\pi)^3\, 2w_{k'}}}\{\varepsilon_\mu(k,\lambda)ik_0'\varepsilon_\nu(k',\lambda')[\partial_0 A^\mu(x),\ A^\nu(x')] - ik_0\varepsilon_i(k,\lambda)\varepsilon_j(k',\lambda')[A \\
&= \int \frac{d^3x d^3x' e^{ikx}e^{-ik'x'}}{\sqrt{(2\pi)^3\, 2w_k\, (2\pi)^3\, 2w_{k'}}}\{\varepsilon_i(k,\lambda)\varepsilon_j(k',\lambda')\,(-i)\,\delta_{ij}^{tr}\,(x-x')\,(ik_0'+ik_0)\}
\end{aligned}$$

Note that

$$\begin{aligned}
l &= \int d^3x d^3x' e^{i\vec{k}\cdot\vec{x}}e^{-i\vec{k}'\cdot\vec{x}'}\varepsilon_i(k,\lambda)\varepsilon_j(k',\lambda')\delta_{ij}^{tr}\,(x-x') = \int d^3x d^3x' e^{i\vec{k}\cdot\vec{x}}e^{-i\vec{k}'\cdot\vec{x}'}\varepsilon_i(k,\lambda)\varepsilon_j(k',\lambda')\int d^3q e^{i\vec{q}\cdot(\vec{x}-} \\
&= \int d^3x d^3x' e^{i(\vec{k}-\vec{q})\cdot\vec{x}}e^{-i(\vec{k}'-\vec{q})\cdot\vec{x}'}\varepsilon_i(k,\lambda)\varepsilon_j(k',\lambda')\int d^3q(\delta_{ij} - \frac{q_iq_j}{q^2}) = \int d^3q\delta^3(\vec{k}-\vec{q})\delta(\vec{k}'-\vec{q})\varepsilon_i(k,\lambda)\varepsilon_j(k \\
&= \delta^3(\vec{k}-\vec{k}')(\delta_{ij} - \frac{k_ik_j}{k^2})\varepsilon_i(k,\lambda)\varepsilon_j(k',\lambda') = \delta^3(\vec{k}-\vec{k}')[\vec{\epsilon}(k,\lambda)\cdot\vec{\epsilon}(k,\lambda') - \frac{1}{k^2}\vec{\epsilon}(k,\lambda)\cdot\vec{k}\left(\vec{\epsilon}(k,\lambda')\cdot\vec{k}\right)] \\
&= \delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}')
\end{aligned}$$

where we have used

$$\vec{\epsilon}(k,\lambda)\cdot\vec{\epsilon}(k,\lambda') = \delta_{\lambda\lambda'}, \qquad \vec{\epsilon}(k,\lambda)\cdot\vec{k} = 0$$

3. In the $\lambda\phi^4$ theory the interacting Lagrangian is of the form,

$$\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$$

For the 2-body elastic scattering we need to compute to second order in λ the following vacuum expectation value

$$\tau^{(2)}\,(y_1,y_2,x_1,x_2) = \frac{(-i)^2}{2!}\int_{-\infty}^{\infty} d^4z_1 d^4z_2\langle 0|T\left(\phi_{in}\,(y_1)\,\phi_{in}\,(y_2)\,\phi_{in}\,(x_1)\,\phi_{in}\,(x_2)\left(\frac{\lambda}{4!}\phi_{in}^4\,(z_1)\right)(\frac{\lambda}{4!}\phi_{in}^4\,(z_2))\right)|0\rangle$$

Use Wick's theorem to write this matrix element in terms of propagators.

4. The Lagrangian for the free fermion field is of the form,

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

Compute the free propagator

$$\int d^4x e^{ipx}\left\langle 0\left|T\left(\psi_\alpha\,(x)\,\bar{\psi}_\beta\,(0)\right)\right|0\right\rangle$$

Solution : The propagator is defined as,

$$\begin{aligned} S_{\alpha\beta}(p) &= \int d^4x e^{ipx} \left\langle 0 \left| T\left(\psi_\alpha(x)\bar{\psi}_\beta(0)\right)\right| 0\right\rangle \\ &= \int d^4x e^{ipx} \left\langle 0 \left| \left(\theta(x_0)\psi_\alpha(x)\bar{\psi}_\beta(0) - \theta(-x_0)\bar{\psi}_\beta(0)\psi_\alpha(x)\right)\right| 0\right\rangle \end{aligned}$$

Note that there is a minus sign for the second term due to the fact that fermion fields anti-commute. Write out the mode expansion,

$$\begin{aligned} \psi_\alpha(x) &= \sum_s \int \frac{d^3p}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2E_p}} \left[b(p,s)u_\alpha(p,s)e^{-ip\cdot x} + d^\dagger(p,s)\upsilon_\alpha(p,s)e^{ip\cdot x}\right] \\ \bar{\psi}_\beta(0) &= \sum_{s'} \int \frac{d^3p'}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2E_p}} \left[b^\dagger(p',s')\bar{u}_\beta(p',s') + d(p',s')\bar{\upsilon}_\beta(p',s')\right] \end{aligned}$$

Then we get

$$\begin{aligned} \left\langle 0 \left|\psi_\alpha(x)\bar{\psi}_\beta(0)\right| 0\right\rangle &= \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E_p}} \int \frac{d^3p'}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2E_p}} u_\alpha(p,s)e^{-ip\cdot x}\bar{u}_\beta(p',s')\left\langle 0 \left| b(p,s)b^\dagger(p',s')\right| 0\right\rangle \\ &= \sum_s \int \frac{d^3p'}{(2\pi)^3} \frac{1}{2E_p} u_\alpha(p',s)\bar{u}_\beta(p',s)e^{-ip'\cdot x} = \int \frac{d^3p'}{(2\pi)^3}\frac{1}{2E_p}\left(\not{p}' + m\right)_{\alpha\beta} e^{-ip'\cdot x} \end{aligned}$$

where we have used

$$\sum_s u_\alpha(p,s)\bar{u}_\beta(p,s) = (\not{p} + m)_{\alpha\beta}$$

Similarly,

$$\left\langle 0 \left|\bar{\psi}_\beta(0)\psi_\alpha(x)\right| 0\right\rangle = \sum_s \int \frac{d^3p'}{(2\pi)^3}\frac{1}{2E_p}\upsilon_\alpha(p',s)\bar{\upsilon}_\beta(p',s)e^{ip'\cdot x} = \int \frac{d^3p'}{(2\pi)^3}\frac{1}{2E_p}\left(\not{p}' - m\right)_{\alpha\beta} e^{ip'\cdot x}$$

and

$$\left\langle 0 \left| \left(\theta(x_0)\psi_\alpha(x)\bar{\psi}_\beta(0) + \theta(-x_0)\bar{\psi}_\beta(0)\psi_\alpha(x)\right)\right| 0\right\rangle = \int \frac{d^3p'}{(2\pi)^3}\frac{1}{2E_p'}[\theta(x_0)\left(\not{p}' + m\right)_{\alpha\beta} e^{-ip'\cdot x} - \theta(-x_0)\left(\not{p}' - m\right)_{\alpha\beta} e^{ip'\cdot}$$

where we have used

$$\sum_s v_\alpha(p,s)\bar{v}_\beta(p,s) = (\not{p} - m)_{\alpha\beta}$$

Note that

$$\frac{1}{2\pi}\int \frac{dp_0}{p_0^2 - E_p^2 + i\varepsilon}e^{-ip_0 t} = \begin{cases} -i\frac{1}{2E_P}e^{-iE_p t} & \text{for} \quad t > 0 \\ -i\frac{1}{2E_P}e^{iE_P t} & \text{for} \quad 0 > t \end{cases}$$

We then get

$$\begin{aligned} \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot x}}{p^2 - m^2 + i\varepsilon}(\not{p} + m) &= -i\int \frac{d^3p}{(2\pi)^3 2E_p}\left[\theta(t)e^{-iE_p t}e^{-i\vec{p}\cdot\vec{x}}\left(E_p\gamma_0 - \vec{\gamma}\cdot\vec{p} + m\right) + \theta(-t)e^{iE_p t}e^{-i\vec{p}\cdot\vec{x}}\left(-E_p\right.\right. \\ &= -i\int \frac{d^3p}{(2\pi)^3 2E_p}\left[\theta(t)e^{-iE_p t}e^{-i\vec{p}\cdot\vec{x}}(\not{p} + m) + \theta(-t)e^{iE_p t}e^{i\vec{p}\cdot\vec{x}}(-\not{p} + m)\right] \\ &= -i\int \frac{d^3p}{(2\pi)^3 2E_p}\left[\theta(t)e^{-ipx}(\not{p} + m) + \theta(-t)e^{-ipx}(-\not{p} + m)\right] \end{aligned}$$

So the fermion propagator is of the form,

$$S_{\alpha\beta}(p) = \int d^4x e^{ipx}\left\langle 0 \left| T\left(\psi_\alpha(x)\bar{\psi}_\beta(0)\right)\right| 0\right\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot x}}{p^2 - m^2 + i\varepsilon}(\not{p} + m)_{\alpha\beta}$$