Quantum Field Theory
Homework 3 Due Wed Nov 18

1. The Dirac equation for free particle is given by,

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi\left(x\right) = 0$$

Under the parity transformation the space-time coordiante transform as

$$x^{\mu} \to x'^{\mu} = (x_0, -x_1, -x_2, -x_3)$$

The Dirac equation in the new coordinate system is of the form,

$$\left(i\gamma^{\mu}\partial'_{\mu} - m\right)\psi'\left(x'\right) = 0$$

Find the relation between $\psi\left(x\right)$ and $\psi'\left(x'\right)$.

2. The left-handed and right-handed components of a Dirac particle are defined by,

$$\psi_L \equiv \frac{1}{2}\left(1 - \gamma_5\right)\psi, \qquad \psi_R \equiv \frac{1}{2}\left(1 + \gamma_5\right)\psi$$

where γ_5 is defined by

$$\gamma_5 \equiv \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

(a) Show that

$$\{\gamma_5, \gamma_\mu\} = 0, \qquad \text{and} \quad \gamma_5^2 = 1$$

(b) Show that ψ_L, ψ_R are eigenstates of γ_5 matrix. What are the eigenvalues?

(c) Are they eigenstates of parity operator?

(d) Write the u spinor in the form,

$$u\left(p, s\right) = N \begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix} \chi_s$$

where N is some normalization constant and χ_s is an arbitrary 2 component spinor. Show that if we choose χ_s to be eigenstate of $\vec{\sigma}\cdot\vec{p}$,

$$\left(\vec{\sigma}\cdot\hat{p}\right)\chi_s = \frac{1}{2}\chi_s$$

then $u\left(p, s\right)$ is an eigenstate of the helicity operator $\lambda = \vec{S}\cdot\hat{p}$ where $\vec{S}$ is the spin operator given by

$$\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

3. Consider a one-dimensional string with length L which satisfies the wave equaiton,

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2}$$

(a) Find the solutions of this wave equation with the boundary conditions,

$$\phi\left(0, t\right) = \phi\left(L, t\right) = 0$$

(b) Find the Lagrangian density which will give this wave equation as the equation of motion.

(c) From the Lagrangian density find the conjugate momenta and impose the quantization conditions. Also find the Hamiltonian.

(d) Find the eigenvalues of the Hamiltonian.

4. Consider the Lagrangian density given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 + J(x)\,\phi, \qquad J(x) \ \ \text{arbitray function}$$

(a) Show that the equation of motion is of the form,

$$\left(\partial^\mu\partial_\mu + \mu^2\right)\phi(x) = J(x)$$

(b) Find the conjugate momenta and impose the quantization conditions.

(c) Find the creation and annihilation operators.

5. Let ϕ be a free scalar field satisfying the field equation,

$$\left(\partial^\mu\partial_\mu + \mu^2\right)\phi(x) = 0$$

(a) Show that the propagator defined by

$$\Delta_F(x-y) \equiv \langle 0\,|T(\phi(x)\,\phi(y))|\,0\rangle = \theta(x_0 - y_0)\,\phi(x)\,\phi(y) + \theta(y_0 - x_0)\,\phi(y)\,\phi(x)$$

can be written as

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \frac{i}{k^2 - \mu^2 + i\varepsilon}$$

(b) Show that the unequal time commutator for these free fields is given by

$$i\Delta(x-y) \equiv [\phi(x)\,, \phi(y)] = \int \frac{d^3k}{(2\pi)^3\, 2\omega_k}\left[e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right]$$

(c) Show that $\Delta(x-y) = 0$ for space-like separation, i.e.

$$\Delta(x-y) = 0, \qquad \text{if} \quad (x-y)^2 < 0$$