# Quantum Electrodynamics

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**Quantum Electrodynamics**

Lagrangian density for QED ,

$$\mathcal{L} = \overline{\psi}(x)\gamma^{\mu}\left(i\partial_{\mu} - eA_{\mu}\right)\psi(x) - m\overline{\psi}(x)\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Equations of motion are

$$\begin{aligned}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) &= eA_{\mu}\gamma^{\mu}\psi \qquad \text{non-linear coupled equations}\\ \partial_{v}F^{\mu\nu} &= e\overline{\psi}\gamma^{\mu}\psi\end{aligned}$$

**Quantization**

Write $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$

$$\begin{aligned}\mathcal{L}_0 &= \overline{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\\ \mathcal{L}_{int} &= -e\overline{\psi}\gamma^{\mu}\psi A_{\mu}\end{aligned}$$

where $\mathcal{L}_0$,free field Lagrangian, $\mathcal{L}_{int}$ is interaction part.

Conjugate momenta for fermion

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_0\psi_{\alpha}\right)} = i\psi_{\alpha}^{\dagger}(x)$$

For em fields choose the gauge

$$\overrightarrow{\nabla}\cdot\overrightarrow{A} = 0$$

Conjugate mometa

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \left(\partial_0 A^i\right)} = -F^{0i} = E^i$$

From equation of motion

$$\partial_\nu F^{0\nu} = e\psi^\dagger \psi \qquad \Longrightarrow \qquad -\nabla^2 A^0 = e\psi^\dagger \psi$$

$A^0$ is not an independent field ,

$$A^0 = e\int d^3x' \, \frac{\psi^\dagger\left(x',t\right)\psi\left(x',t\right)}{4\pi|\overrightarrow{x'} - \overrightarrow{x}|} = e\int \frac{d^3x' \rho\left(x',t\right)}{|\overrightarrow{x} - \overrightarrow{x'}|}$$

Commutation relations

$$\begin{aligned}\left\{\psi_\alpha\left(\overrightarrow{x},t\right), \psi_\beta^\dagger\left(\overrightarrow{x'},t\right)\right\} &= \delta_{\alpha\beta}\delta^3\left(\overrightarrow{x} - \overrightarrow{x'}\right) \qquad \left\{\psi_\alpha\left(\overrightarrow{x},t\right), \psi_\beta\left(\overrightarrow{x'},t\right)\right\} = \ldots = 0 \\ \left[\dot{A}_i\left(\overrightarrow{x},t\right), A_j\left(\overrightarrow{x'},t\right)\right] &= i\delta_{ij}^{tr}\left(\overrightarrow{x} - \overrightarrow{x'}\right)\end{aligned}$$

where

$$\delta_{ij}^{tr}(\vec{x} - \vec{y}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)$$

Commutators involving $A_0$

$$\left[A_0\left(\overrightarrow{x},t\right), \psi_\alpha\left(\overrightarrow{x'},t\right)\right] = e\int \frac{d^3x''}{4\pi|\overrightarrow{x} - \overrightarrow{x''}|}\left[\psi^\dagger\left(\overrightarrow{x''},t\right)\psi\left(\overrightarrow{x''},t\right), \psi_\alpha\left(\overrightarrow{x'},t\right)\right] = -\frac{e}{4\pi}\frac{\psi_\alpha\left(\overrightarrow{x'},t\right)}{|\overrightarrow{x} - \overrightarrow{x'}|}$$

Hamiltonian density

$$\begin{aligned}\mathcal{H} &= \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \psi_\alpha\right)}\dot{\psi}_\alpha + \frac{\partial \mathcal{L}}{\partial\left(\partial_0 A^k\right)}\dot{A}_k - \mathcal{L} \\ &= \psi^\dagger\left(-i\overrightarrow{\alpha}\cdot\overrightarrow{\nabla}+\beta m\right)\psi + \frac{1}{2}\left(\overrightarrow{E}^2+\overrightarrow{B}^2\right) + \overrightarrow{E}\cdot\overrightarrow{\nabla}A_0 + e\overline{\psi}\gamma^\mu\psi A_\mu\end{aligned}$$

and

$$H = \int d^3x\mathcal{H} = \int d^3x\{\psi^\dagger\left[\overrightarrow{\alpha}\cdot(-i\overrightarrow{\nabla}-e\overrightarrow{A})+\beta m\right]\psi + \frac{1}{2}\left(\overrightarrow{E}^2+\overrightarrow{B}^2\right)\}$$

$A_0$ does not appear in the interaction,
But if we write

$$\overrightarrow{E} = \overrightarrow{E_l} + \overrightarrow{E_t} \qquad \text{where} \quad \overrightarrow{E_l} = -\overrightarrow{\nabla}A_0 \qquad , \overrightarrow{E_t} = -\frac{\partial \overrightarrow{A}}{\partial t}$$

Then

$$\frac{1}{2}\int d^3x\left(\overrightarrow{E}^2+\overrightarrow{B}^2\right) = \frac{1}{2}\int d^3x\overrightarrow{E_l}^2 + \int d^3x\left(\overrightarrow{E_t}^2+\overrightarrow{B}^2\right)$$

longitudinal part is

$$\frac{1}{2}\int d^3x\overrightarrow{E_l}^2 = \frac{e}{4\pi}\int d^3x d^3y\frac{\rho\left(\overrightarrow{x},t\right)\rho\left(\overrightarrow{y},t\right)}{|\overrightarrow{x}-\overrightarrow{y}|} \qquad \text{Coulomb interaction}$$

Without classical solutions, can not do mode expansion to get creation and annihilation operators We can only do perturbation theory.

Recall that the free field part $\overrightarrow{A}_0$ satisfy massless Klein-Gordon equation

$$\Box \overrightarrow{A}^{(0)} = 0$$

The solution is

$$\overrightarrow{A}^{(0)}(\vec{x},t) = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_\lambda \overrightarrow{\epsilon}(k,\lambda)[a(k,\lambda)e^{-ikx} + a^+(k,\lambda)e^{ikx}] \qquad w = k_0 = |\overrightarrow{k}|$$

$$\vec{\epsilon}(k,\lambda),\ \lambda = 1,2 \quad \text{with} \qquad \vec{k}\cdot\vec{\epsilon}(k,\lambda) = 0$$

Standard choice

$$\vec{\epsilon}(k,\lambda)\cdot\vec{\epsilon}(k,\lambda') = \delta_{\lambda\lambda'}, \quad \vec{\epsilon}(-k,1) = -\vec{\epsilon}(k,1), \quad \vec{\epsilon}(-k,2) = \vec{\epsilon}(-k,2)$$

It is convienent to write the mode expansion as,

$$A_\mu(\vec{x},t) = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_\lambda \epsilon_\mu(k,\lambda)[a(k,\lambda)e^{-ikx} + a^+(k,\lambda)e^{ikx}]$$

where

$$\epsilon_\mu(k,\lambda) = (0,\vec{\epsilon}(k,\lambda))$$

## Photon Propagator

Feynman propagatpr for photon is

$$\begin{aligned} iD_{\mu\nu}(x,x') &= \left\langle 0 \left| T\left(A_\mu(x) A_\nu(x')\right)\right| 0 \right\rangle \\ &= \theta(t-t')\left\langle 0 \left| A_\mu(x) A_\nu(x')\right| 0 \right\rangle + \theta(t'-t)\left\langle 0 \left| A_\nu(x') A_\mu(x)\right| 0 \right\rangle \end{aligned}$$

From mode expansion,

$$\begin{aligned} \left\langle 0 \left| A_\mu(x) A_\nu(x')\right| 0 \right\rangle &= \int \frac{d^3k d^3k'}{(2\pi)^3\sqrt{2\omega_k 2\omega_{k'}}} \sum_{\lambda,\lambda'} \epsilon_\mu(k,\lambda)\epsilon_\nu(k',\lambda') \left\langle 0 \left| [a(k,\lambda)e^{-ikx}] a^+(k',\lambda')e^{ik'x'} \right| 0 \right\rangle \\ &= \int \frac{d^3k d^3k'}{(2\pi)^3 2\omega_k} \sum_{\lambda,\lambda'} \epsilon_\mu(k,\lambda)\epsilon_\nu(k',\lambda')\ \delta^3(k-k')\, e^{-ikx+ik'x'} \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda,\lambda'} \epsilon_\mu(k,\lambda)\epsilon_\nu(k,\lambda') e^{-ik(x-x')} \end{aligned}$$

Note that

$$\frac{1}{2\pi}\int \frac{dk_0}{k_0^2-\omega^2+i\varepsilon} e^{-ik_0(t-t')} = \begin{cases} -i\frac{1}{2\omega} e^{-i\omega(t-t')} & \text{for} \quad t > t' \\ -i\frac{1}{2\omega} e^{i\omega(t-t')} & \text{for} \quad t' > t \end{cases}$$

We then get

$$\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x'-x)}}{k^2+i\varepsilon} = -i\int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[\theta(t-t')e^{-ik(x-x')} + \theta(t-t')e^{ik(x-x')}\right]$$

and

$$\begin{aligned}\left\langle 0\left|T\left(A_{\mu}\left(x\right)A_{\nu}\left(x'\right)\right)\right|0\right\rangle &= \int\frac{d^3k}{(2\pi)^3 2\omega_k}\sum_{\lambda,\lambda'}\epsilon_{\mu}(k,\lambda)\epsilon_{\nu}(k,\lambda')[\theta(t-t')e^{-ik(x-x')}+\theta(t-t')e^{ik(x-x')}]\\ &= i\int\frac{d^4k}{\left(2\pi\right)^4}\frac{e^{-ik\cdot\left(x'-x\right)}}{k^2+i\varepsilon}\sum_{\lambda=1}^{2}\epsilon_{\nu}(k,\lambda)\epsilon_{\mu}(k,\lambda)=iD_{\mu\nu}\left(x,x'\right)\end{aligned}$$

Or

$$D_{\mu\nu}\left(x,x'\right)=\int\frac{d^4k}{\left(2\pi\right)^4}\frac{e^{-ik\cdot\left(x'-x\right)}}{k^2+i\varepsilon}\sum_{\lambda=1}^{2}\epsilon_{\nu}(k,\lambda)\epsilon_{\mu}(k,\lambda)$$

polarization vectors $\epsilon_{\mu}(k,\lambda)$, $\lambda=1,2$ are perpendicular to each other. Add 2 more unit vectors to form a complete set

$$\eta^{\mu}=\left(1,0,0,0\right), \qquad \hat{k}^{\mu}=\frac{k^{\mu}-\left(k\cdot\eta\right)\eta^{\mu}}{\sqrt{\left(k\cdot\eta\right)^2-k^2}}$$

completeness relation is then,

$$\begin{aligned}\sum_{\lambda=1}^{2}\epsilon_{\nu}(k,\lambda)\epsilon_{\mu}(k,\lambda) &= -g_{\mu\nu}-\eta_{\mu}\eta_{\nu}-\hat{k}_{\mu}\hat{k}_{\nu}\\ &= -g_{\mu\nu}-\frac{k_{\mu}k_{\nu}}{\left(k\cdot\eta\right)^2-k^2}+\frac{\left(k\cdot\eta\right)\left(k_{\mu}\eta_{\nu}+\eta_{\mu}k_{\nu}\right)}{\left(k\cdot\eta\right)^2-k^2}-\frac{k^2\eta_{\mu}\eta_{\nu}}{\left(k\cdot\eta\right)^2-k^2}\end{aligned}$$

If we define propagator in momentum space as

$$D_{\mu\nu}\left(x,x'\right)=\int\frac{d^4k}{\left(2\pi\right)^4}e^{-ik\cdot\left(x'-x\right)}D_{\mu\nu}\left(k\right)$$

then

$$D_{\mu\nu}\left(k\right)=\frac{1}{k^2+i\varepsilon}\left[-g_{\mu\nu}-\frac{k_\mu k_\nu}{\left(k\cdot\eta\right)^2-k^2}+\frac{\left(k\cdot\eta\right)\left(k_\mu\eta_\nu+\eta_\mu k_\nu\right)}{\left(k\cdot\eta\right)^2-k^2}-\frac{k^2\eta_\mu\eta_\nu}{\left(k\cdot\eta\right)^2-k^2}\right]$$

terms proportional to $k_\mu$ will not contribute to physical processes and the last term is of the form $\delta_{\mu 0}\ \delta_{\nu 0}$ will be cancelled by the Coulom interaction..

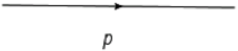
## Feynman rule in QED

The interaction Hamiltonian is ,

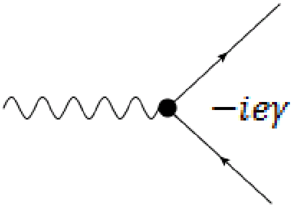
$$H_{int} = e \int d^3x \bar{\psi}\gamma^{\mu}\psi A_{\mu}$$

The Feynman propagators, vertices and external wave functions are given below.

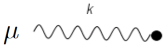

$$\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}$$



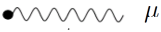

$$\frac{i}{\not{p} - m}$$



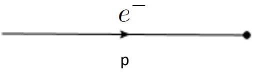
$$-ie\gamma^{\mu}$$



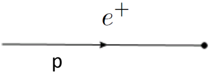

$$\varepsilon_{\mu}(k)$$



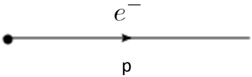

$$\varepsilon_{\mu}(k)$$



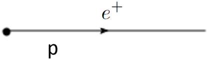

$u(p, s)$




$\overline{v}(p, s)$




$\overline{u}(p, s)$



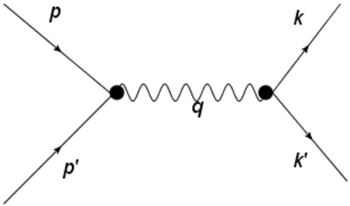

$v(p, s)$

$e^+e^- \to \mu^+\mu^-$

**Total Cross Section**

momenta for this reaction

$$e^+ (p') + e^- (p) \to \mu^+ (k') + \mu^- (k)$$




Use Feynman rule to write the matrix element as

$$\begin{aligned} M(e^+e^- \quad &\to \quad \mu^+\mu^-) = \bar{v}(p', s') \left(-ie\gamma^\mu\right) u\left(p, s\right) \left(\frac{-ig_{\mu\nu}}{q^2}\right) \bar{u}(k', r') \left(-ie\gamma^\nu\right) v\left(k, r\right) \\ &= \quad \frac{ie^2}{q^2} \bar{v}(p', s')\gamma^\mu u\left(p, s\right) \bar{u}(k', r')\gamma_\mu v\left(k, r\right) \end{aligned}$$

where $q = p + p'$. Notet that electron vertex have property,

$$q_\mu \bar{v}(p')\gamma^\mu u\left(p\right) = \left(p + p'\right)_\mu \bar{v}(p')\gamma^\mu u\left(p\right) = \bar{v}(p') \left(\not{p} + \not{p}'\right) u\left(p\right) = 0$$

This shows the term proportional to photon momentum $q^\mu$ will not contribute in the physical processes. For cross section, we need $M^*$ which contains factor $\left(\bar{v}\gamma^\mu u\right)^*$

$$\left(\bar{v}\gamma^\mu u\right)^* = u^\dagger\left(\gamma^\mu\right)^\dagger\left(\gamma_0\right)^\dagger v = u^\dagger\gamma_0\gamma^\mu v = \bar{u}\gamma^\mu v$$

More generally,

$$\left(\bar{v}\Gamma u\right)^* = \bar{u}\bar{\Gamma}v, \qquad \text{with} \quad \bar{\Gamma} = \gamma^0\Gamma^\dagger\gamma^0$$

It is easy to see

$$\bar{\gamma}_\mu = \gamma_\mu$$

$$\overline{\gamma_\mu\gamma_5} = -\gamma_\mu\gamma_5$$

$$\overline{\not{a}\not{b}\cdots\not{p}} = \not{p}\cdots\not{b}\not{a}$$

unpolarized cross section which requires the spin sum,

$$\sum_s u_\alpha\left(p,s\right)\bar{u}_\beta\left(p,s\right) = \left(\not{p}+m\right)_{\alpha\beta}$$

$$\sum_s v_\alpha\left(p,s\right)\bar{v}_\beta\left(p,s\right) = \left(\not{p}-m\right)_{\alpha\beta}$$

This can be seen as follows.

$$\begin{aligned}\sum_s u_\alpha\left(p,s\right)\bar{u}_\beta\left(p,s\right) &= (E+m)\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\end{pmatrix}\sum_s\chi_s\chi_s^\dagger\begin{pmatrix} 1 & -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\end{pmatrix} = (E+m)\begin{pmatrix} 1 & -\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} & \frac{-\vec{p}^2}{(E+m)^2}\end{pmatrix} \\ &= \begin{pmatrix} E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E+m\end{pmatrix} = \not{p}+m\end{aligned}$$

Similarly for the $v-$spinor,

$$\begin{aligned}\sum_s v_\alpha(p,s)\,\bar{v}_\beta(p,s) &= (E+m)\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\\ 1\end{pmatrix}\chi_s\chi_s^\dagger\begin{pmatrix}\frac{\vec{\sigma}\cdot\vec{p}}{E+m} & -1\end{pmatrix} = (E+m)\begin{pmatrix}\frac{\vec{p}^2}{(E+m)^2} & -\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} & -1\end{pmatrix}\\ &= \begin{pmatrix}E-m & -\vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & -(E+m)\end{pmatrix} = \not{p}-m\end{aligned}$$

A typical calculation is,

$$\begin{aligned} & \sum_{s,s'} \bar{v}_\alpha(p',s') (\gamma^\mu)_{\alpha\beta} u_\beta(p,s) \bar{u}_\rho(p,s) (\gamma^\nu)_{\rho\sigma} v_\sigma(p,s) \\ = & \sum_{s'} \bar{v}_\alpha(p',s') (\gamma^\mu)_{\alpha\beta} (\not{p}+m)_{\beta\rho} (\gamma^\nu)_{\rho\sigma} v_\sigma(p,s) \\ = & (\gamma^\mu)_{\alpha\beta} (\not{p}+m)_{\beta\rho} (\gamma^\nu)_{\rho\sigma} (\not{p}-m)_{\sigma\alpha} \\ = & Tr\left[\gamma^\mu (\not{p}+m) \gamma^\nu (\not{p}-m)\right] \end{aligned}$$

trace of product of $\gamma$ matrices.

$$Tr(\gamma^\mu) = 0$$

$$Tr(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

$$Tr\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = 4\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}\right)$$

$$\begin{aligned} & Tr(\not{a}_1\not{a}_2\cdots\not{a}_n) \\ = & (a_1\cdot a_2)\, Tr(\not{a}_3\cdots\not{a}_n) - (a_1\cdot a_3)\, Tr(\not{a}_2\cdots\not{a}_n) + \cdots + (a_1\cdot a_n)\, Tr\left(\not{a}_2\not{a}_3\cdots\not{a}_{n-1}\right), \qquad n \text{ even} \\ = & 0 \qquad n \text{ odd} \end{aligned}$$

With these tools

$$\frac{1}{4}\sum_{spin}\left|M(e^+e^-\to\mu^+\mu^-)\right|^2 = \frac{e^4}{q^4} Tr\left[(\not{p}'-m_e)\gamma^\mu(\not{p}+m_e)\gamma^\nu\right] Tr\left[(\not{k}'+m_\mu)\gamma_\mu(\not{k}+m_\mu)\gamma^\nu\right]$$

for energies $\gg m_\mu$.

$$\frac{1}{4}\sum_{spin'}\left|M(e^+e^-\to\mu^+\mu^-)\right|^2 = 8\frac{e^4}{q^4}\left[(p\cdot k)\left(p'\cdot k'\right)+(p'\cdot k)\left(p\cdot k'\right)\right]$$

In center of mass,

$$p_\mu = (E,0,0,E)\,,\qquad p'_\mu = (E,0,0,-E)$$

$$k_\mu = \left(E,\vec{k}\right),\qquad k'_\mu = \left(E,-\vec{k}\right),\qquad \text{with } \vec{k}\cdot\hat{z} = E\cos\theta$$

If we set $m_\mu = 0$, $E = \left|\vec{k}\right|$ and

$$q^2 = (p+p')^2 = 4E^2,\qquad p\cdot k = p'\cdot k' = E^2\left(1-\cos\theta\right),$$

$$p'\cdot k = p\cdot k' = E^2\left(1+\cos\theta\right)$$

Then

$$\begin{aligned}\frac{1}{4}\sum_{spin'}|M|^2 &= \frac{8e^4}{16E^4}\left[E^4\left(1-\cos\theta\right)^2+E^4\left(1+\cos\theta\right)^2\right]\\ &= e^4\left(1+\cos^2\theta\right)\end{aligned}$$

Note that under the parity $\theta\to\pi-\theta$. this matrix element conserves the parity

The cross section is

$$d\sigma = \frac{1}{I}\frac{1}{2E}\frac{1}{2E}(2\pi)^4\delta^4(p+p'-k-k')\frac{1}{4}\sum_{spin'}|M|^2\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}$$

use the $\delta-$function to carry out integrations . introduce the quantity $\rho$, called the **phase space**, given by

$$\begin{aligned}\rho &= \int(2\pi)^4\delta^4(p+p'-k-k')\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'} \\ &= \frac{d\Omega}{32\pi^2}\end{aligned}$$

The flux factor is

$$I = \frac{1}{E_1E_2}\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2} = \frac{1}{E^2}2E^2 = 2$$

The differential crossection is then

$$d\sigma = \frac{1}{2}\frac{1}{4E^2}\left(\frac{1}{4}\sum_{spin'}|M|^2\right)\frac{d\Omega}{32\pi^2}$$

Or

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}\left(1+\cos^2\theta\right)$$

where $\alpha = \dfrac{e^2}{4\pi}$ is the fine structure constant. The total cross section is

$$\sigma\left(e^+e^- \to \mu^+\mu^-\right) = \frac{\alpha^2\pi}{3E^2}$$

Or

$$\sigma\left(e^+e^- \to \mu^+\mu^-\right) = \frac{4\alpha^2\pi}{3s} \qquad \text{with} \qquad s = (p_1 + p_2)^2 = 4E^2$$

$\underline{e^+e^- \to hadrons}$

One of the interesting processess in $e^+e^-$ collider is the reaction

$$e^+e^- \to hadrons$$

According to QCD, theory of strong interaciton, this processes will go through

$$e^+e^- \to q\bar{q}$$

and then $q\bar{q}$ trun into hadrons. Since coupling of $\gamma$ to $q\bar{q}$ differs from the coupling to $\mu^+\mu^-$ only in their charges cross section for $q\bar{q}$ as

$$\sigma\left(e^+e^- \to q\bar{q}\right) = 3\left(Q_q^2\right)\frac{4\alpha^2\pi}{3s} = 3\left(Q_q^2\right)\sigma\left(e^+e^- \to \mu^+\mu^-\right)$$

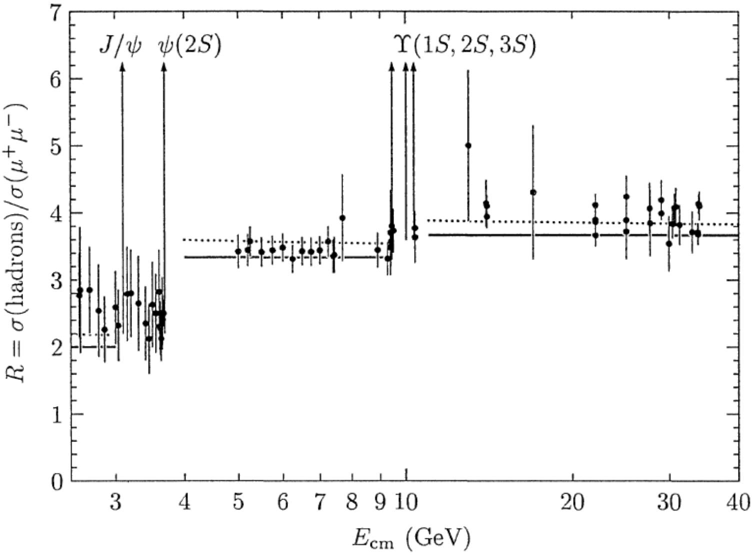
$Q_q$ is electric charge of quark $q$. The factor of 3 because each quark has 3 colors. Then

$$\frac{\sigma\left(e^+e^- \to hadrons\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)} = 3\left(\sum_i Q_i^2\right)$$

Summation is over quarks which are allowed by the avaliable energies. e. g., for energy below the the charm quark only $u, d$, and $s$ quarks should be included,

$$\frac{\sigma\left(e^+e^- \to hadrons\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)} = 3\left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right] = 2$$

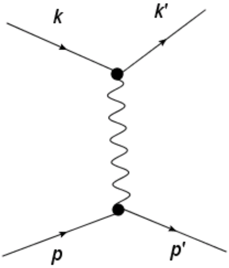
which is not far from the reality.



7
6
5
4
3
2
1
0
$J/\psi$
$\psi(2S)$
$\Upsilon(1S, 2S, 3S)$
$R = \sigma(\mathrm{hadrons})/\sigma(\mu^+\mu^-)$
3
4
5
6
7
8
9
10
20
30
40
$E_{\mathrm{cm}}$ (GeV)

ep → ep,

$$e\,(k) + p\,(p) \longrightarrow e\,(k') + p\,(p')$$




Proton has strong interaction. First consider proton has no strong interaction and include strong interaction later. The lowest order contribution is ,

$$\begin{aligned} M(e+p \quad \rightarrow \quad e+p) &= \bar{u}(p',s')\,(-ie\gamma^{\mu})\,u\,(p,s)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}(k',r')\,(-ie\gamma^{\nu})\,u\,(k,r) \\ &= \frac{ie^2}{q^2}\,\bar{u}(p',s')\gamma^{\mu}u\,(p,s)\,\bar{u}(k',r')\gamma_{\mu}u\,(k,r) \end{aligned}$$

where $q = k - k'$. For unploarized cross section, sum over the spins ,

$$\frac{1}{4}\sum_{spin}|M(e+p\rightarrow e+p)|^2 = \frac{e^4}{q^4} Tr\left[\left(\not{p}'+M\right)\gamma^{\mu}\left(\not{p}+M\right)\gamma^{\nu}\right] Tr\left[\left(\not{k}'+m_e\right)\gamma_{\mu}\left(\not{k}+m_e\right)\gamma^{\nu}\right]$$

Again neglect $m_e$. Compute the traces

$$Tr\left[\not{k}'\gamma_{\mu}\not{k}\gamma^{\nu}\right] = 4\left[k'^{\mu}k^{\nu} - g^{\mu\nu}\left(k\cdot k'\right) + k^{\mu}k'^{\nu}\right]$$

$$Tr\left[\left(\not{p}'+M\right)\gamma^{\mu}\left(\not{p}+M\right)\gamma^{\nu}\right] = 4\left[p'^{\mu}p^{\nu} - g^{\mu\nu}\left(p\cdot p'\right) + p^{\mu}p'^{\nu}\right] + 4M^2 g^{\mu\nu}$$

Then

$$\frac{1}{4}\sum_{spin}|M(e+p\rightarrow e+p)|^2 = \frac{e^4}{q^4}\left\{8\left[(p\cdot k)\left(p'\cdot k'\right) + (p'\cdot k)\left(p\cdot k'\right)\right] - 8M^2\left(k\cdot k'\right)\right\}$$

Use laboratoy frame

$$p_\mu = (M, 0, 0, 0)\,, \qquad k_\mu = \left(E, \vec{k}\right), \qquad k'_\mu = \left(E', \vec{k'}\right)$$

Then

$$p \cdot k = ME, \qquad p.k' = ME', \qquad k \cdot k' = EE'\left(1 - \cos\theta\right)$$

$$p' \cdot k' = \left(p + k - k'\right).k' = p.k' + k \cdot k', \qquad p' \cdot k = \left(p + k - k'\right).k = p.k - k \cdot k'$$

$$q^2 = \left(k - k'\right)^2 = -2k \cdot k' = -2EE'\left(1 - \cos\theta\right)$$

Differential cross section is

$$d\sigma = \frac{1}{I}\frac{1}{2p_0}\frac{1}{2k_0}(2\pi)^4\delta^4(p + k - p' - k')\frac{1}{4}\sum_{spin'}|M|^2\frac{d^3p'}{(2\pi)^3 2p'_0}\frac{d^3k'}{(2\pi)^3 2k'_0}$$

The phase space is

$$\begin{aligned} \rho &= \int (2\pi)^4\delta^4(p + k - p' - k')\frac{d^3p'}{(2\pi)^3 2p'_0}\frac{d^3k'}{(2\pi)^3 2k'_0} \\ &= \frac{1}{4\pi^2}\int \delta\left(p_0 + k_0 - p'_0 - k'_0\right)\frac{d^3k'}{2p'_0 2k'_0} \end{aligned} \tag{1}$$

where

$$p'_0 = \sqrt{M^2 + \left(\vec{p} + \vec{k} - \vec{k'}\right)^2} = \sqrt{M^2 + \left(\vec{k} - \vec{k'}\right)^2}$$

Use the momenta in lab frame,

$$\begin{aligned}\rho &= \frac{1}{4\pi^2}\int \delta\left(M+E-p_0'-E'\right)\frac{k'^2 dk' d\Omega}{2p_0' 2E'} \\ &= \frac{1}{4\pi^2}\int \delta\left(M+E-p_0'-E'\right)\frac{d\Omega E' dE'}{p_0'}\end{aligned}$$

Let

$$x = -E + p_0' + E'$$

Then

$$dx = dE'(1+\frac{dp_0'}{dE'}) = dE'\left(\frac{p_0' + E' - E\cos\theta}{p_0'}\right)$$

and

$$\rho = \frac{1}{4\pi^2}\int \delta\left(x-M\right)\frac{d\Omega E' dx}{\left(p_0'+E'-E\cos\theta\right)} = \frac{1}{4\pi^2}\frac{d\Omega E'}{M+E\left(1-\cos\theta\right)}$$

From the argument of the $\delta-$function we get the relation, $M = x = -E + p_0' + E'$
From momentum conservation

$$p_0'^2 = M^2 + \left(\vec{k} - \vec{k'}\right)^2 = M^2 + E^2 + E'^2 - 2EE'\cos\theta$$

and from energy conservation

$$p_0'^2 = \left(M+E-E'\right)^2 = M^2 + E^2 + E'^2 - 2EE' + 2ME - 2ME'$$

Comparing these 2 equations we can solve for $E'$,

$$E' = \frac{ME}{E\left(1-\cos\theta\right)+M} = \frac{E}{1+\left(\frac{2E}{M}\right)\sin^2\frac{\theta}{2}}$$

The phase space is then

$$\rho = \frac{d\Omega}{4\pi^2}\frac{ME}{\left(M+E\left(1-\cos\theta\right)\right)^2} = \frac{d\Omega}{4\pi^2}\frac{E'^2}{ME}$$

The flux factor is

$$I = \frac{1}{ME}p\cdot k = 1$$

The differential cross section is then

$$d\sigma = \frac{1}{I}\frac{1}{2p_0}\frac{1}{2k_0}(2\pi)^4\delta^4(p+k-p'-k')\frac{1}{4}\sum_{spin'}|M|^2\frac{d^3p'}{(2\pi)^3 2p_0'}\frac{d^3k'}{(2\pi)^3 2k_0'}$$

Or

$$\frac{d\sigma}{d\Omega} = \frac{1}{4ME}\frac{1}{4\pi^2}\frac{E'^2}{ME}\frac{1}{4}\sum_{spin'}|M|^2 = \left(\frac{E'}{E}\right)^2\frac{1}{16\pi^2M^2}\frac{e^4}{q^4}\left\{8\left[(p\cdot k)\left(p'\cdot k'\right)+(p'\cdot k)\left(p\cdot k'\right)\right]-8M^2\left(k\cdot k'\right)\right\}$$

It is staightforward to get

$$\begin{aligned} & \left[(p\cdot k)\left(p'\cdot k'\right)+(p'\cdot k)\left(p\cdot k'\right)\right]-M^2\left(k\cdot k'\right) \\ = \quad & 2EE'M^2\left[\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right] \end{aligned}$$

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \left(\frac{E'}{E}\right)^2\frac{\alpha^2}{M^2}\frac{1}{\left(4EE'\sin^2\frac{\theta}{2}\right)^2}2EE'M^2\left[\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right] \\ &= \frac{\alpha^2}{4}\frac{E'}{E^3}\frac{1}{\sin^4\frac{\theta}{2}}\left[\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right] \end{aligned}$$

Or

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{4E^2}\frac{1}{\sin^4\frac{\theta}{2}}\frac{\left[\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]}{\left[1+\left(\frac{2E}{M}\right)\sin^2\frac{\theta}{2}\right]}$$

**Strong interaction**.
Use the fact that the $\gamma pp$ interaction is local to parametrize the $\gamma pp$ matrix element as

$$\left\langle p'\left|J_\mu\right|p\right\rangle=\bar{u}(p',s')\left[\gamma^\mu F_1\left(q^2\right)+\frac{i\sigma_{\mu\nu}q^\nu}{2M}F_2\left(q^2\right)\right]u\left(p,s\right)\qquad\text{with}\qquad q=p-p' \tag{2}$$

Lorentz covariance and current conservation have been used. Another useful relation is the Gordon decomposition

$$\bar{u}(p')\gamma_\mu u(p) = \bar{u}(p')\left[\frac{(p+p')^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\right]u(p)$$

This can be derived as follows. From Dirac equation

$$(\not{p}-m)u(p) = 0, \qquad \bar{u}(p')(\not{p}'-m) = 0$$

and

$$\bar{u}(p')\gamma^\mu(\not{p}-m)u(p) = 0, \qquad \bar{u}(p')(\not{p}'-m)\gamma^\mu u(p) = 0$$

Adding these equations,

$$\begin{aligned} 2m\bar{u}(p')\gamma_\mu u(p) &= \bar{u}(p')\left(\gamma_\mu\not{p}+\not{p}'\gamma_\mu\right)u(p) = \bar{u}(p')\left(p^\nu\gamma_\mu\gamma_\nu + p'^\nu\gamma_\nu\gamma_\mu\right)u(p) \\ &= \bar{u}(p')\left(p^\nu(\frac{1}{2}\left\{\gamma_\mu,\gamma_\nu\right\} + \frac{1}{2}\left[\gamma_\mu,\gamma_\nu\right]) + p'^\nu(\frac{1}{2}\left\{\gamma_\mu,\gamma_\nu\right\} - \frac{1}{2}\left[\gamma_\mu,\gamma_\nu\right])\right)u(p) \end{aligned}$$

From this we get

$$\bar{u}(p')\gamma_\mu u(p) = \bar{u}(p')\left[\frac{(p+p')^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p'-p)_\nu}{2m}\right]u(p)$$

$F_1(q^2)$, charge form factor
$F_2(q^2)$, magnetic form factor .
Note that $F_1(q^2) = 1$ and $F_2(q^2) = 0$ correspond to point particle.
The charge form factor satifies the condition $F_1(0) = 1$. From

$$Q|p\rangle = |p\rangle$$

we get

$$\left\langle p' \left| Q \right| p \right\rangle = \left\langle p' | p \right\rangle = 2E \left(2\pi\right)^3 \delta^3 \left(\vec{p} - \vec{p}'\right)$$

On the other hand from Eq(2) we see that

$$\begin{aligned}\left\langle p'\left|Q\right|p\right\rangle &= \int d^3x \left\langle p'\left|J_0\left(x\right)\right|p\right\rangle = \int d^3x \left\langle p'\left|J_0\left(0\right)\right|p\right\rangle e^{i\left(p'-p\right)\cdot x} \\ &= \left(2\pi\right)^3 \delta^3\left(\vec{p}-\vec{p}'\right)\bar{u}(p',s')\gamma_0 u\left(p,s\right)F_1\left(0\right) \\ &= 2E\left(2\pi\right)^3\delta^3\left(\vec{p}-\vec{p}'\right)F_1\left(0\right)\end{aligned}$$

compare two equations $\Longrightarrow F_1\left(0\right)=1$. To gain more insight, write $Q$ in terms of charge density

$$Q=\int d^3x\rho\left(x\right)=\int d^3xJ_0\left(x\right)$$

Then

$$\left\langle p'\left|J_0\left(x\right)\right|p\right\rangle = e^{iq\cdot x}\left\langle p'\left|J_0\left(0\right)\right|p\right\rangle = e^{iq\cdot x}F_1\left(q^2\right)\bar{u}(p',s')\gamma_0 u\left(p,s\right)$$

$F_1\left(q^2\right)$ is the Fourier transform of charge density distribution i.e.

$$F_1\left(q^2\right)\sim\int d^3x\rho\left(x\right)e^{-i\vec{q}\cdot\vec{x}}$$

Expand $F_1\left(q^2\right)$ in powers of $q^2$,

$$F_1\left(q^2\right)=F_1\left(0\right)+q^2F_1'\left(0\right)+\cdots$$

$F_1\left(0\right)$ is total charge and $F_1'\left(0\right)$ is related to the charge radius.
Calulate cross section as before,

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{4E^2}\frac{\left[\cos^2\frac{\theta}{2}\left(\frac{1}{1-q^2/4M^2}\right)\left[G_E^2-\left(q^2/4M^2\right)G_M^2\right]-\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}G_M^2\right]}{\sin^4\frac{\theta}{2}\left[1+\left(\frac{2E}{M}\right)\sin^2\frac{\theta}{2}\right]}$$

where

$$G_E = F_1 + \frac{q^2}{4M^2} F_2$$

$$G_M = F_1 + F_2$$

Experimentally, $G_E$ and $G_M$ have the form,

$$G_E\left(q^2\right) \approx \frac{G_M\left(q^2\right)}{\varkappa_p} \approx \frac{1}{\left(1 - q^2/0.7Gev^2\right)^2} \tag{3}$$

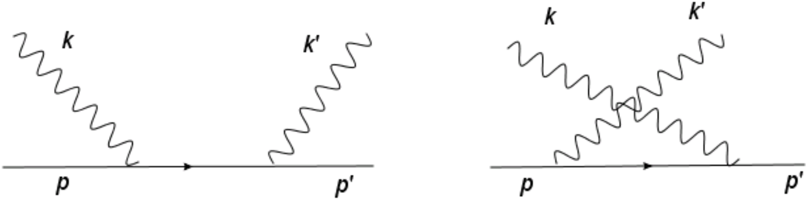
where $\varkappa_p = 2.79$ magnetic moment of the proton. If proton were point like, we would have $G_E\left(q^2\right)$ $= G_M\left(q^2\right) = 1$

Dependence of $q^2$ in Eq(3) $\Longrightarrow$ proton has a structure. For large $q^2$ the elastic cross section falls off rapidly as $G_E \approx G_M \sim q^{-4}$.

**Compton Scattering**

$$\gamma(k) + e(p) \longrightarrow \gamma(k') + e(p')$$

Two diagrams contribute,




The amplitude is given by

$$M(\gamma e \longrightarrow \gamma e) = \bar{u}(p')(-ie\gamma^{\mu})\varepsilon'_{\mu}(k') \frac{i}{\not{p} + \not{k} - m} (-ie\gamma^{\nu}) \varepsilon_{\nu}(k) u(p)$$
$$+\bar{u}(p')(-ie\gamma^{\mu})\varepsilon_{\mu}(k) \frac{i}{\not{p} - \not{k}' - m} (-ie\gamma^{\nu}) \varepsilon'_{\nu}(k') u(p)$$

Note that if write the amplitude as

$$M = \varepsilon'_{\mu}(k') M^{\mu}$$

Then we have

$$k'_\mu M^\mu = -ie^2 [\bar{u}(p') \not{k}' \frac{1}{\not{p} + \not{k} - m} \gamma^\nu \varepsilon_\nu (k) u(p) + \bar{u}(p') \gamma^\mu \varepsilon_\mu (k) \frac{i}{\not{p} - \not{k}' - m} \not{k}' u(p)]$$

Using the relation

$$\not{k}' = (\not{p}' + \not{k}' - m) - (\not{p}' - m) = (\not{p} - m) - (\not{p} - \not{k}' - m)$$

we get

$$k'_\mu M^\mu = -ie^2 [\bar{u}(p') \gamma^\nu \varepsilon_\nu (k) u(p) - \bar{u}(p') \gamma^\mu \varepsilon_\mu (k) u(p)] = 0$$

Similarly, we can show that if we repalce the polarization $\varepsilon_\mu (k)$ by $k_\mu$ the amplitude also vanishes. Using this relation we can simplify the polarization sum for the photon as follows. Let us take polarizations to be

$$\varepsilon_\mu (k, 1) = (0, 1, 0, 0), \qquad \varepsilon_\mu (k, 2) = (0, 0, 1, 0), \qquad \text{with} \quad k_\mu = (k, 0, 0, k),$$

Then the polarization sum is

$$\sum_\lambda \left| \varepsilon_\mu (k, \lambda) M^\mu \right|^2 = \left| M^1 \right|^2 + \left| M^2 \right|^2$$

But from $k_\mu M^\mu = 0$, we get

$$M^0 = M^3, \qquad \Longrightarrow \qquad \left| M^0 \right|^2 = \left| M^3 \right|^2$$

Thus

$$\sum_\lambda \left| \varepsilon_\mu (k, \lambda) M^\mu \right|^2 = \left| M^1 \right|^2 + \left| M^2 \right|^2 + \left| M^3 \right|^2 - \left| M^0 \right|^2 = -g_{\mu\nu} M^\mu M^{*\nu}$$

This is the same as the repalcement

$$\sum_\lambda \varepsilon_\mu (k, \lambda) \varepsilon_\mu (k, \lambda) \longrightarrow -g_{\mu\nu}$$

Put the $\gamma$- matrices in the numerator,

$$M = -ie^2 \varepsilon'_\mu \varepsilon_\nu \left[ \bar{u}(p') \gamma^\mu \frac{\not{p} + \not{k} + m}{2p \cdot k} \gamma^\nu u\,(p) + \bar{u}(p') \gamma^\nu \frac{\not{p} - \not{k}' + m}{-2p \cdot k'} \gamma^\mu u\,(p) \right]$$

Using the relations,

$$(\not{p} + m)\, \gamma^\nu u\,(p) = 2p^\nu u\,(p)\,,$$

we get

$$M = -ie^2 \bar{u}(p') \left[ \frac{\not{\varepsilon}' \not{k} \not{\varepsilon} + 2\,(p \cdot \varepsilon)\, \not{\varepsilon}'}{2p \cdot k} + \frac{-\not{\varepsilon} \not{k}' \not{\varepsilon}' + 2\,(p \cdot \varepsilon)\, \not{\varepsilon}'}{-2p \cdot k'} \right] u\,(p)$$

The photon polarizations are,

$$\varepsilon_\mu = \left(0, \vec{\varepsilon}\right), \qquad \text{with} \quad \vec{\varepsilon}\cdot\vec{k} = 0, \qquad \varepsilon'_\mu = \left(0, \vec{\varepsilon}'\right), \qquad \text{with} \quad \vec{\varepsilon}'\cdot\vec{k}' = 0,$$

Lab frame , $p_\mu = (m, 0, 0, 0)$, $\Longrightarrow$ $(p\cdot\varepsilon) = (p\cdot\varepsilon') = 0$ and

$$M = -ie^2\overline{u}(p')\left[\frac{\not{\varepsilon}'\not{k}\not{\varepsilon}}{2p\cdot k} + \frac{\not{\varepsilon}\not{k}'\not{\varepsilon}'}{2p\cdot k'}\right]u\left(p\right)$$

Summing over spin of the electron

$$\frac{1}{2}\sum_{spin}|M|^2 = e^4\,Tr\left\{\left(\not{p}' + m\right)\left[\frac{\not{\varepsilon}'\not{k}\not{\varepsilon}}{2p\cdot k} + \frac{\not{\varepsilon}\not{k}'\not{\varepsilon}'}{2p\cdot k'}\right](\not{p} + m)\left[\frac{\not{\varepsilon}'\not{k}\not{\varepsilon}}{2p\cdot k} + \frac{\not{\varepsilon}\not{k}'\not{\varepsilon}'}{2p\cdot k'}\right]\right\}$$

The cross section is given by

$$d\sigma = \frac{1}{I}\frac{1}{2p_0}\frac{1}{2k_0}(2\pi)^4\delta^4(p + k - p' - k')\frac{1}{4}\sum_{spin'}|M|^2\frac{d^3p'}{(2\pi)^3 2p'_0}\frac{d^3k'}{(2\pi)^3 2k'_0}$$

phase space

$$\rho = \int(2\pi)^4\delta^4(p + k - p' - k')\frac{d^3p'}{(2\pi)^3 2p'_0}\frac{d^3k'}{(2\pi)^3 2k'_0}$$

is exactly the same as the case for *ep* scattering and the result is

$$\rho = \frac{d\Omega}{4\pi^2}\frac{\omega'^2}{m\omega}$$

It is straightforward to compute the trace with result,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 4\left(\varepsilon\cdot\varepsilon'\right)^2 - 2\right]$$

This is **Klein-Nishima** relation. In the limit $\omega \to 0$,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{m^2}\left(\varepsilon\cdot\varepsilon'\right)^2$$

here $\dfrac{\alpha}{m}$ is classical electron radius.

For unpolarized cross section, sum over polarization of photon,

$$\sum_{\lambda\lambda'} \left[\varepsilon(k,\lambda)\cdot\varepsilon'(k',\lambda')\right]^2 = \sum_{\lambda\lambda'} \left[\vec{\varepsilon}(k,\lambda)\cdot\vec{\varepsilon}'(k',\lambda')\right]^2$$

Since $\vec{\varepsilon}(k,1)$, $\vec{\varepsilon}(k,2)$ and $\vec{k}$ form basis in 3-dimension, completeness relation is

$$\sum_{\lambda} \varepsilon_i(k,\lambda)\,\varepsilon_j(k,\lambda) = \delta_{ij} - \hat{k}_i\hat{k}_j$$

Then

$$\sum_{\lambda\lambda'} \left[\vec{\varepsilon}(k,\lambda)\cdot\vec{\varepsilon}'(k',\lambda')\right]^2 = \left(\delta_{ij} - \hat{k}_i\hat{k}_j\right)\left(\delta_{ij} - \hat{k}'_i\hat{k}'_j\right) = 1 + \cos^2\theta$$

where $\hat{k}\cdot\hat{k}' = \cos\theta$. The cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]$$

The total cross section,

$$\sigma = \frac{\pi\alpha^2}{m^2}\int_{-1}^{1} dz\left\{\frac{1}{\left[1+\frac{\omega}{m}(1-z)\right]^3} + \frac{1}{\left[1+\frac{\omega}{m}(1-z)\right]} - \frac{1-z^2}{\left[1+\frac{\omega}{m}(1-z)\right]^2}\right\}$$

At low energies, $\omega \to 0$, we

$$\sigma = \frac{8\pi\alpha^2}{3m^2}$$

and at high energies

$$\sigma = \frac{\pi\alpha^2}{\omega m}\left[\ln\frac{2\omega}{m} + \frac{1}{2} + O\left(\frac{m}{\omega}\ln\frac{m}{\omega}\right)\right]$$