QFT Perturbation Theory

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Interaction Theory

As an illustration, take electromagnetic interaction. Lagrangian density is

$$\mathcal{L} = \overline{\psi}(x)\gamma^{\mu}\left(i\partial_{\mu} - eA_{\mu}\right)\psi(x) - m\overline{\psi}(x)\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

The combination

$$D_{\mu} = \left(\partial_{\mu} + eA_{\mu}\right)$$

is the covariant derivative.
Equations of motion

$$\begin{aligned} \left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) &= eA_{\mu}\gamma^{\mu}\psi \qquad \text{non-linear coupled equations} \\ \partial_{\upsilon}F^{\mu\nu} &= e\overline{\psi}\gamma^{\mu}\psi \end{aligned}$$

No exact solutions are known. Expansion in terms of Fourier components,

$$\psi\left(\vec{x}, t\right) = \sum_{s}\int \frac{d^{3}p}{\sqrt{(2\pi)^{3}2E_{p}}}\left[b(p,s,t)\,u(p,s)\,e^{-i\vec{p}\cdot\vec{x}} + d^{\dagger}(p,s,t)\,\upsilon(p,s)\,e^{i\vec{p}\cdot\vec{x}}\right]$$

Then the operators b and $d^{\dagger}$ time dependent and the time evolution is controlled by the interaction terms.

Quantization

Write $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$

$$\begin{aligned} \mathcal{L}_0 &= \overline{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ \mathcal{L}_{int} &= -e\overline{\psi}\gamma^{\mu}\psi A_{\mu} \end{aligned}$$

where $\mathcal{L}_0$,free field part, while $\mathcal{L}_{int}$ interaction.
Conjugate momenta

$$\frac{\partial \mathcal{L}}{\partial \left(\partial_0 \psi_\alpha\right)} = i\psi_\alpha^\dagger \left(x\right)$$

For electromagnetic fields, choose the gauge

$$\overrightarrow{\nabla} \cdot \overrightarrow{A} = 0 \ , \qquad \text{Radiaiton Gauge}$$

then

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \left(\partial_0 A^i\right)} = -F^{0i} = E^i$$

From equation of motion

$$\partial_\nu F^{0\nu} = e\psi^\dagger \psi \qquad \Longrightarrow \qquad -\nabla^2 A^0 = e\psi^\dagger \psi$$

We can't take A_0 to be zero but A^0 can be expressed in terms of other field,

$$A^0 = e \int d^3x' \, \frac{\psi^\dagger \left(x', t\right) \psi \left(x', t\right)}{4\pi |\overrightarrow{x'} - \overrightarrow{x}|} = e \int \frac{d^3x' \rho \left(x', t\right)}{4\pi |\overrightarrow{x} - \overrightarrow{x'}|}$$

Commutation relations

$$\begin{aligned} \left\{\psi_\alpha \left(\overrightarrow{x}, t\right), \psi_\beta^\dagger \left(\overrightarrow{x'}, t\right)\right\} &= \delta_{\alpha\beta}\delta^3 \left(\overrightarrow{x} - \overrightarrow{x'}\right) \qquad \left\{\psi_\alpha \left(\overrightarrow{x}, t\right), \psi_\beta \left(\overrightarrow{x'}, t\right)\right\} = ... = 0 \\ \left[\dot{A}_i \left(\overrightarrow{x}, t\right), A_j \left(\overrightarrow{x'}, t\right)\right] &= i\delta_{ij}^{tr} \left(\overrightarrow{x} - \overrightarrow{x'}\right) \end{aligned}$$

Commutators with A_0 ,

$$\left[A_0\left(\overrightarrow{x}, t\right), \psi_\alpha(\overrightarrow{x'}, t)\right] = e\int \frac{d^3x''}{4\pi|\overrightarrow{x} - \overrightarrow{x''}|}\left[\psi^\dagger(\overrightarrow{x''}, t)\psi(\overrightarrow{x''}, t), \psi_\alpha\left(\overrightarrow{x'}, t\right)\right]$$

$$= -\frac{e}{4\pi}\frac{\psi_\alpha\left(\overrightarrow{x'}, t\right)}{|\overrightarrow{x} - \overrightarrow{x'}|}$$

We have used the relationWe have used the relation

$$\nabla^2 \frac{1}{4\pi|\overrightarrow{x} - \overrightarrow{x'}|} = -\delta^3\left(\overrightarrow{x} - \overrightarrow{x'}\right)$$

This can be seen as follows. In the spherical coordinates the Laplacian is of the form,

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

From this we get

$$\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{(-1)}{r^2}\right) = 0, \qquad \text{if} \quad r \neq 0$$

To deal with the singularity at $r \to 0$, consider the integral

$$\int d^3x\ \nabla^2\left(\frac{1}{r}\right) = \int d^3x\ \vec{\nabla}\cdot[\vec{\nabla}\left(\frac{1}{r}\right)] = \oint d\vec{S}\cdot\ [\vec{\nabla}\left(\frac{1}{r}\right)]$$

We can take the surface to be the surface of a sphere with radius a centered around the origin so that $d\overrightarrow{S} = a^2 \hat{r} d\Omega$. Using $\overrightarrow{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$, we get

$$\nabla^2 \frac{1}{4\pi|\overrightarrow{x} - \overrightarrow{x'}|} = -\delta^3\left(\overrightarrow{x} - \overrightarrow{x'}\right)$$

Hamiltonian

$$H = \int d^3x \mathcal{H} = \int d^3x \{\psi^\dagger \left[\overrightarrow{\alpha} \cdot (-i\overrightarrow{\nabla} - e\overrightarrow{A}) + \beta m\right] \psi + \frac{1}{2}\left(\overrightarrow{E}^2 + \overrightarrow{B}^2\right)\}$$

No A_0 in the interaction. If we write

$$\overrightarrow{E} = \overrightarrow{E_l} + \overrightarrow{E_t} \qquad \text{where} \quad \overrightarrow{E_l} = -\overrightarrow{\nabla} A_0 \qquad , \overrightarrow{E_t} = -\frac{\partial \overrightarrow{A}}{\partial t}$$

Then

$$\frac{1}{2}\int d^3x \left(\overrightarrow{E}^2 + \overrightarrow{B}^2\right) = \frac{1}{2}\int d^3x \overrightarrow{E_l}^2 + \int d^3x \left(\overrightarrow{E_t}^2 + \overrightarrow{B}^2\right)$$

The longitudinal part is

$$\frac{1}{2}\int d^3x \overrightarrow{E_l}^2 = -\frac{1}{2}\int d^3x A_0 \nabla^2 A_0 = \frac{e^2}{4\pi}\int d^3x d^3y \frac{\rho\left(\overrightarrow{x}, t\right)\rho\left(\overrightarrow{y}, t\right)}{|\overrightarrow{x} - \overrightarrow{y}|}$$

Perturbation Theory

Can't solve the classical equations of motion. Can't do mode expansion to introduce a and $a^{\dagger}$.The only approximation we can do in field theory is the perturbation theory.
We will now set up the framework for the perturbation.

Physical states

In high energy physics, we study the scattering processes.
Assume interactions all short-range, far away from interaction region, particles propagate as free particles.
Choose the physical states to be eigensates of energy momentum operators,

$$P_{\mu}|\Psi\rangle = p_{\mu}|\Psi\rangle$$

Satisfy requirements;

1. eigenvalues p_{μ} all in forward light cone,

$$p^2 = p_{\mu}p^{\mu} \geqslant 0, \qquad p_0 \geqslant 0$$

2. non-degenerate Lorentz invariant ground state $|0>$ with zero energy ,

$$p^0|0\rangle = 0, \qquad \Longrightarrow \vec{p}|0\rangle = 0$$

3. There exists stable single particle states $|\vec{p_i}\rangle$ with $p_i^2 = m_i^2$ for each stable particle.
4. vaccum and one particle states form discrete spectra in p^{μ}

assume interactions do not violently the spectrum of states. there is no room to describe bound states

Spectral Representation

Eventhough it is difficult to get exact results in the interacting field theories, it is still possible to get some useful relation independent of the details of the interaction. Here we will discuss one such example of spectral representation of commutator or propagator of interacting scalar fields.

We write the vacuum expectation value of commutator of scalar fields as

$$i\Delta'(x,y) = \langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \langle 0 | (\phi(x)\phi(y) - \phi(y)\phi(x)) | 0 \rangle$$

Inserting a complete set of eigenstates with definite energy and mometum between two field operators, we get for the first term

$$\sum_n \langle 0 | \phi(x) | n \rangle \langle n | \phi(y) | 0 \rangle = \sum_n e^{-ip_n \cdot x} e^{ip_n \cdot y} \langle 0 | \phi(0) | n \rangle \langle n | \phi(0) | 0 \rangle$$

where we have use the relation

$$\langle 0 | \phi(x) | n \rangle = \left\langle 0 \left| e^{iP\cdot x} \phi(0) e^{-iP\cdot x} \right| n \right\rangle = e^{-ip_n \cdot x} \langle 0 | \phi(0) | n \rangle$$

The commutator function is then

$$i\Delta'(x,y) = \sum_n \langle 0 | \phi(0) | n \rangle \langle n | \phi(0) | 0 \rangle \left[e^{-ip_n \cdot (x-y)} - e^{ip_n \cdot (x-y)} \right] = i\Delta'(x-y)$$

We see that the commutator function is function of the difference $x-y$ rather than x, y separately. It is useful to group togather states with same p_n by introducing the identity,

$$1 = \int d^4 q \delta^4 (q - p_n)$$

so that we can write

$$\begin{aligned}\Delta'(x-y) &= \frac{-i}{(2\pi)^3}\int d^4q\sum_n (2\pi)^3\,\delta^4(q-p_n)\,|\langle 0|\phi(0)|\,n\rangle|^2\,[e^{-ip_n\cdot(x-y)}-e^{ip_n\cdot(x-y)}]\\ &= \frac{-i}{(2\pi)^3}\int d^4q\rho(q)\,[e^{-ip_n\cdot(x-y)}-e^{ip_n\cdot(x-y)}]\end{aligned}$$

where

$$\rho(q)=\sum_n (2\pi)^3\,\delta^4(q-p_n)\,|\langle 0|\phi(0)|\,n\rangle|^2$$

is usually called the spectral function. It is clear that $\rho(q)$ is a Lorentz scalar and can be written as

$$\rho(q)=\rho\left(q^2\right)\theta(q_0)$$

due to the fact that all physical momenta are non-zero only in the forward light-cone. Thus $\rho\left(q^2\right)$ vanishes for $q^2<0$ and is positive semi-definite for $q^2>0$.It is convienent to write the commutator function as

$$\begin{aligned}\Delta'(x-y) &= \frac{-i}{(2\pi)^3}\int d^4q\rho\left(q^2\right)\theta(q_0)\,[e^{-iq\cdot(x-y)}-e^{iq\cdot(x-y)}]\\ &= \int d\sigma^2\rho\left(\sigma^2\right)\int d^4q\delta\left(q^2-\sigma^2\right)\varepsilon(q_0)\,e^{-iq\cdot(x-y)}\\ &= \int_0^\infty d\sigma^2\rho\left(\sigma^2\right)\Delta\left(x-y,\sigma^2\right)\end{aligned} \tag{1}$$

where

$$\Delta\left(x-y,\sigma^2\right)=\int d^4q\delta\left(q^2-\sigma^2\right)\varepsilon(q_0)\,e^{-iq\cdot(x-y)}$$

is the commutator function for free scalar field with mass σ^2.In this Eq (1), the commutator function of the interacting field is expersed as integral over free field commutatotor function with different masses is called the spectral representation of the commutator function.
Exactly the same procedure can be applied to the propagator function to get

$$\Delta'_F(x-y) = -i\langle 0|T(\phi(x)\phi(y))|0\rangle = \int_0^\infty d\sigma^2 \rho(\sigma^2)\,\Delta_F(x-y,\sigma^2)$$

Or in momentum space

$$\Delta'_F(p^2) = \int_0^\infty d\sigma^2 \rho(\sigma^2)\,\frac{1}{p^2-\sigma^2+i\varepsilon}$$

where

$$\Delta'_F(p^2) = \int d^4x \Delta'_F(x)\, e^{-ip\cdot x}, \qquad \Delta_F(p^2) = \int d^4x \Delta_F(x)\, e^{-ip\cdot x}$$

In-fields and in-states—asymptotic conditions

Consider

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{\mu_0^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

Eq. of motion

$$\left(\Box + \mu_0^2\right)\phi = j\left(x\right) = \frac{\lambda}{3!}\phi^3$$

conjugate momenta

$$\pi\left(x\right) = \frac{\partial \mathcal{L}}{\partial_0 \phi} = \partial_0 \phi$$

Commutation relations

$$\left[\pi\left(x,t\right),\phi\left(y,t\right)\right] = -i\delta^3\left(x-y\right) \qquad \left[\pi\left(x,t\right),\pi\left(y,t\right)\right] = \left[\phi\left(x,t\right),\phi\left(y,t\right)\right] = 0$$

At $t = -\infty$, $\phi_{in}\left(x\right)$ creates free particle with physical mass μ.

$$\left(\Box + \mu^2\right)\phi_{in}\left(x\right) = 0$$

we allow physical mass μ to be different from μ_0 .
Assume that $\phi_{in}\left(x\right)$ transforms same way as $\phi\left(x\right)$. In particular,

$$\left[p_\mu, \phi_{in}\left(x\right)\right] = -i\partial_\mu \phi_{in}\left(x\right)$$

$\phi_{in}\left(x\right)$ creates one particle state from vacuum.
Expand $\phi_{in}\left(x\right)$ in terms of free solution of Klein-Gordon equation,

$$\phi_{in}\left(x\right) = \int d^3k\left[a_{in}\left(k\right)f_k\left(x\right) + a_{in}^\dagger\left(k\right)f_k^*\left(x\right)\right] \qquad f_k\left(x\right) = \frac{1}{\sqrt{\left(2\pi\right)^3 2w_k}}e^{-ik\cdot x}$$

Invert this expansion

$$a_{in}(k) = i \int d^3x f_k^*(x) \overleftrightarrow{\partial_0} \phi_{in}(x)$$

We also have

$$\left[p^\mu, a_{in}(k)\right] = -k^\mu a_{in}(k), \qquad \left[p^\mu, a_{in}^\dagger(k)\right] = k^\mu a_{in}^\dagger(k)$$

States are defined by

$$|k_1, in\rangle = \sqrt{(2\pi)^3 2w_k} a_{in}^\dagger(k) |0\rangle$$

$$|k_1, k_2, ...k_n|in\rangle = \left[\prod_i \sqrt{(2\pi)^3 2w_{k_i}} a_{in}^\dagger(k_i)\right] |0\rangle$$

Relation between $\phi_{in}(x)$ and $\phi(x)$

The field equations for interacting fields

$$\left(\Box + \mu_0^2\right)\phi(x) = j(x)$$

Or

$$\left(\Box + \mu^2\right)\phi(x) = j(x) + \delta\mu^2\phi(x) = \widetilde{j(x)} \qquad \delta\mu^2 = \mu^2 - \mu_0^2$$

and for the in-fields,

$$\left(\Box + \mu^2\right)\phi_{in}(x) = 0$$

Formally relates $\phi(x)$ to $\phi_{in}(x)$ by Green's function,

$$\sqrt{z}\phi_{in}(x) = \phi(x) - \int d^4y \triangle_{ret}\left(x-y,\mu^2\right)\widetilde{j(y)}$$

where

$$\left(\Box_x + \mu^2\right)\triangle_{ret}\left(x-y,\mu^2\right) = \delta^4(x-y) \ , \ \triangle_{ret}\left(x-y,\mu^2\right) = 0 \quad \text{for} \quad x_0 < y_0$$

is the retarded Green's function. The factor $\sqrt{z}$ is included to permit the normalization of ϕ_{in} to unit amplitude for its matrix element to create one-particle state from vacuum. This suggests that

$$\phi(x) \rightarrow \sqrt{z}\phi_{in}(x) \qquad as \quad x_0 \rightarrow -\infty, \tag{2}$$

because the retarded Green's function vanishes as $x_0 \rightarrow -\infty$. This relation which relates 2 operators can be viewed as strong convergence relation.It turns out that this leads to contradiction. To see this take matrix element of Eq (**??**), between vacuum and one-particle state

$$\sqrt{z}\left\langle 0\left|\phi_{in}(x)\right|p\right\rangle = \left\langle 0\left|\phi(x)\right|p\right\rangle - \int d^4y \triangle_{ret}\left(x-y,\mu^2\right)\left\langle 0\left|\tilde{j}(y)\right|p\right\rangle$$

The second term vanishes because

$$\left\langle 0\left|\tilde{j}\left(y\right)\right|p\right\rangle = \left(\Box+\mu^{2}\right)\left\langle 0\left|\phi\left(x\right)\right|p\right\rangle = \left(\Box+\mu^{2}\right)e^{-ip\cdot x}\left\langle 0\left|\phi\left(0\right)\right|p\right\rangle = \left(-p^{2}+\mu^{2}\right)e^{-ip\cdot x}\left\langle 0\left|\phi\left(0\right)\right|p\right\rangle = 0$$

So we get

$$\sqrt{z}\left\langle 0\left|\phi_{in}\left(x\right)\right|p\right\rangle = \left\langle 0\left|\phi\left(x\right)\right|p\right\rangle$$

and we interpret $\sqrt{z}$ as the probabilty amplitude for the interacting field to produce one-particle from the vacuum. Using the mode expansion for the free field $\phi_{in}\left(x\right)$, we get

$$\left\langle 0\left|\phi_{in}\left(x\right)\right|p\right\rangle = \int\frac{d^{3}k}{\sqrt{\left(2\pi\right)^{3}2w_{k}}}e^{-ik\cdot x}\left\langle 0\left|a_{in}\left(k\right)\right|p\right\rangle = \int\frac{d^{3}k}{\sqrt{\left(2\pi\right)^{3}2w_{k}}}e^{-ik\cdot x}\sqrt{\left(2\pi\right)^{3}2w_{p}}\delta^{3}\left(p-k\right) = e^{-ip\cdot x}$$

and

$$\left\langle 0\left|\phi\left(0\right)\right|p\right\rangle = \sqrt{z}\left\langle 0\left|\phi_{in}\left(0\right)\right|p\right\rangle$$

Its contribution to the spectral function is

$$\rho\left(q^{2}\right)_{1particle} = \int\frac{d^{3}p}{\left(2\pi\right)^{3}2w_{p}}\left(2\pi\right)^{3}\delta^{4}\left(p-q\right)\left|\left\langle 0\left|\phi\left(0\right)\right|p\right\rangle\right|^{2} = Z\delta\left(p_{0}-q_{0}\right)\frac{1}{2\omega_{p}} = Z\delta\left(q^{2}-\mu^{2}\right)\theta\left(q_{0}\right)$$

Put this into the spectral representation in Eq (1)

$$\Delta'\left(x-y\right) = Z\Delta\left(x-y,\mu^{2}\right) + \int_{4\mu^{2}}^{\infty}d\sigma^{2}\rho\left(\sigma^{2}\right)\Delta\left(x-y,\sigma^{2}\right)$$

The lower limit in the integral corresponds to the threshold for all states other than the one particle state. Differentiate this with respect to x_0 and set $x_0 = y_0$ to get the canonical equal time commutator,

$$\lim_{x_0 \longrightarrow y_0} \frac{\partial}{\partial x_0} i\Delta'(x-y) = \langle 0 | [\partial_0 \phi(x, x_0), \phi(y, x_0)] | 0 \rangle = -i\delta^3(x-y)$$
$$= \lim_{x_0 \longrightarrow y_0} \frac{\partial}{\partial x_0} [iZ\Delta(x-y, \mu^2) + \int_{4\mu^2}^{\infty} d\sigma^2 \rho(\sigma^2) \Delta(x-y, \sigma^2)]$$
$$= -i\delta^3(x-y)[Z + \int_{4\mu^2}^{\infty} d\sigma^2 \rho(\sigma^2)]$$

where we have used

$$\lim_{x_0 \longrightarrow y_0} \frac{\partial}{\partial x_0} \Delta(x-y, \mu^2) = -i\delta^3(x-y)$$

This follows from the fact that $\Delta(x-y, \sigma^2)$ simply corresponds to commutator for free scalar fields. Thus we get the relation

$$1 = Z + \int_{4\mu^2}^{\infty} d\sigma^2 \rho(\sigma^2)$$

which implies the inequality

$$0 \leq Z < 1 \tag{3}$$

The inequality $Z < 1$ is due to the positivity of the spectral function $\rho(q^2)$. Now we use this inequality to show that the asymptotic condition in Eq (2) will lead to contradiction. Recall that both the interacting field $\phi(x)$ and the free field $\phi_{in}(x)$ satisfy the same equal time commutation relation,

$$[\partial_0 \phi(x, x_0), \phi(y, x_0)] = -i\delta^3(x-y), \qquad [\partial_0 \phi_{in}(x, x_0), \phi_{in}(y, x_0)] = -i\delta^3(x-y)$$

These relations can be consistent with the asymptotic relation $\phi(x) \to \sqrt{z}\phi_{in}(x) \qquad as \quad x_0 \to -\infty$, in Eq (2) only if $Z = 1$, which contradicts the inequality in Eq (3).

Correct asymptotic condition (Lehmann, Symanzik, and Zimmermann**)**

Let $|\alpha\rangle, |\beta\rangle$ be any two normalizable states, $\phi^f(t)$ is defined

$$\phi^f(t) \equiv i\int d^3x f_k^*(\overrightarrow{x}, t)\overleftrightarrow{\partial_0}\phi(\overrightarrow{x}, t) \quad \text{with} \quad \left(\Box + \mu^2\right) f = 0$$

$f_k(\overrightarrow{x}, t)$ is an arbitrary *normalizable* solution to Klein-Gordon equation.

The correct asymptotic condition is

$$\lim_{x_0 \to -\infty} \langle\alpha|\phi^f(t)|\beta\rangle = \sqrt{z}\langle\alpha|\phi^f_{in}(t)|\beta\rangle \quad \text{with} \quad \phi^f_{in}(t) = i\int d^3x f^*(\overrightarrow{x}, t)\overleftrightarrow{\partial_0}\phi_{in}(\overrightarrow{x}, t)$$

Out fields and out states

Similar procedure applies to ϕ_{out}

$$\left(\Box + \mu_0^2\right) \phi_{out}(x) = 0$$

$$\phi_{out}(x) = \int d^3k \left[a_{out}(k) f_k(x) + a_{out}^\dagger(k) f_k^*(x)\right],$$

$$\left[p^\mu, a_{out}^\dagger(k)\right] = -k^\mu a_{out}^\dagger(k)$$

Asymptotic condition

$$\lim_{t\to\infty} \langle\alpha|\phi^f(t)|\beta\rangle = \sqrt{z}\langle\alpha|\phi_{out}^f(t)|\beta\rangle$$

S-matrix

Scattering processes: start n non-interacting particles.They interact when close to each other. After interaction, m particles seperate

Initial state

$$|p_1, p_2, ..., p_n, in\rangle = |\alpha, in\rangle$$

Final state

$$|p_1', p_2', ..., p_m', out\rangle = |\beta, out\rangle$$

S-matrix

$$S_{\beta\alpha} \equiv \langle\beta, out|\alpha, in\rangle$$

Introduce S-operator

$$\langle\beta, out| \equiv \langle\beta, in|S \qquad \langle\beta, out|S^{-1} = \langle\beta, in|$$

then the $S - matrix$ element can be written as a matrix element of $S-$operator between 2 $in-states$,

LSZ reduction
set up the framework to compute $S_{\beta\alpha}$.
Consider

$$S_{\beta,\alpha p} = \langle \beta, out | \alpha, p, in \rangle$$

Using creation operator for the $in - state$,

$$\begin{aligned} S_{\beta,\alpha p} &= \langle \beta, out | \alpha, p, in \rangle = \sqrt{(2\pi)^3 2w_p} \langle \beta, out | a^{\dagger}_{in}(p) | \alpha, in \rangle \\ &= \sqrt{(2\pi)^3 2w_p} \langle \beta, out | a^{\dagger}_{out}(p) | \alpha, in \rangle + \langle \beta, out | \left[a^{\dagger}_{in}(p) - a^{\dagger}_{out}(p) \right] | \alpha, in \rangle] \\ &= N \left[\langle \beta - p, out | \alpha, in \rangle - i \langle \beta, out | \int d^3 x f_p(x) \overleftrightarrow{\partial_0} \left[\phi_{in}(x) - \phi_{out}(x) \right] \alpha, in \rangle \right] \end{aligned}$$

Here $\langle \beta - p, out|$ is state $\langle \beta, out|$ by removing a particle with momentum $\overrightarrow{p}$ and $N = \sqrt{(2\pi)^3 2w_p}$
Use the symptotic conditions

$$\langle \alpha | \phi_{in}(x) | \beta \rangle = \frac{1}{\sqrt{z}} \lim_{t \to -\infty} \langle \alpha | \phi(x) | \beta \rangle, \qquad \langle \alpha | \phi_{out}(x) | \beta \rangle = \frac{1}{\sqrt{z}} \lim_{t \to \infty} \langle \alpha | \phi(x) | \beta \rangle$$

and the identity

$$\left(\lim_{x_0 \to \infty} - \lim_{x_0 \to -\infty} \right) \int d^3 x g_1(x) \overleftrightarrow{\partial_0} g_2(x) = \int_{-\infty}^{\infty} d^4 x \left[g_1(x) \partial_0^2 g_2(x) - \partial_0^2 g_1(x) g_2(x) \right]$$

$$\int d^3x f_p(x) \overleftrightarrow{\partial_0} [\phi_{in}(x) - \phi_{out}(x)] = \int d^4x \left[\partial_0^2 f_p(x)\phi(x) - f_p(x)\partial_0^2\phi(x)\right]$$
$$= -\int d^4x f_p(x) \quad (\Box + \mu^2)\phi(x)$$

we get the **reduction formula**,

$$\langle \beta, out|\alpha, p, in\rangle = N\langle \beta - p, out|\alpha, in\rangle + \frac{i}{\sqrt{z}}\int e^{-ip\cdot x} d^4x \left(\Box + \mu^2\right)\langle \beta, out|\phi(x)|\alpha, in\rangle$$

where we have used $\partial_0^2 f_p(x) = \left(\partial_i^2 - \mu^2\right) f_p(x)$ and carry out the integration by parts.
To remove a particle with momentum p' from β

$$\langle \beta, out|\phi(x)|\alpha, in\rangle = \langle \gamma p', out|\phi(x)|\alpha, in\rangle = N\langle \gamma, out|a_{out}(p')\phi(x)|\alpha, in\rangle$$

$$= [\langle \gamma, out|\phi(x) a_{in}(p')|\alpha, in\rangle - \langle \gamma, out|(a_{out}(p')\phi(x) - \phi(x) a_{in}(p')|\alpha, in\rangle]$$

$$= \langle \gamma, out|\phi(x)|\alpha - p', in\rangle - i\int d^3y \langle \gamma, out|(\phi_{out}(y)\phi(x) - \phi(x)\phi_{in}(y))|\alpha, in\rangle \overleftrightarrow{\partial_0} f^*_{p'}(y)$$

$$= \langle \gamma, out|\phi(x)|\alpha - p', in\rangle$$
$$-\frac{i}{\sqrt{z}}\int d^3y \left(\lim_{y_0\to\infty} - \lim_{y_0\to-\infty}\right)\langle \gamma, out|\left(T\left(\phi(y)\phi(x)\right)\right)|\alpha, in\rangle \overleftrightarrow{\partial_0} f^*_{p'}(y)$$

same procedure as before

$$\langle \beta, out|\phi(x)|\alpha, in\rangle = \{\langle \gamma, out|\phi(x)|\alpha - p', in\rangle\} + \frac{i}{\sqrt{z}} \int d^4y \langle \gamma, out| T(\phi(y)\phi(x))|\alpha, in\rangle \left(\overleftarrow{\Box}_y + \mu^2\right) e^{ip\cdot x}$$

remove all particles from "in" and "out" state

$$\langle p_1, ..., p_n, out|q_1, ..., q_m, in\rangle = \left(\frac{i}{\sqrt{z}}\right)^{m+n} \prod_{i=1}^{m}\prod_{j=1}^{n} \int d^4x_i d^4y_j e^{-iq_i x_i} \left(\overrightarrow{\Box}_x + \mu^2\right) \langle 0| T(\phi(y_1)...\phi(y_m)\phi(x_1)...\phi(x_m))|0\rangle \left(\overleftarrow{\Box}_{y_j} + \mu^2\right) e^{ip_j\cdot x_j}$$

for all $p_j \neq q_i$remove all particles from "in" and "out" state

$$\langle p_1, ..., p_n, out|q_1, ..., q_m, in\rangle = \left(\frac{i}{\sqrt{z}}\right)^{m+n} \prod_{i=1}^{m}\prod_{j=1}^{n} \int d^4x_i d^4y_j e^{-iq_i x_i} \left(\overrightarrow{\Box}_x + \mu^2\right) \langle 0| T(\phi(y_1)...\phi(y_m)\phi(x_1)...\phi(x_m))|0\rangle \left(\overleftarrow{\Box}_{y_j} + \mu^2\right) e^{ip_j\cdot x_j}$$

for all $p_j \neq q_i$

In and Out fields for Fermions

generalization to fermions.
in-field

$$\psi_{in}(x) = \int d^3p \sum_s \left[b_{in}(p,s)\, U_{p,s}(x) + d^\dagger_{in}(p,s)\, V_{p,s}(x) \right]$$

where

$$U_{p,s}(x) = \frac{1}{\sqrt{(2\pi)^3 2E_p}} u(p,s)\, e^{-ip\cdot x} \qquad V_{p,s}(x) = \frac{1}{\sqrt{(2\pi)^3 2E_p}} \upsilon(p,s)\, e^{ip\cdot x}$$

Inversion

$$\begin{aligned} b_{in}(p,s) &= \int d^3x U^\dagger_{p,s}(x)\, \psi_{in}(x) \qquad d_{in}(p,s) = \int d^3x \psi^\dagger_{in}(x)\, V_{p,s}(x) \\ b^\dagger_{in}(p,s) &= \int d^3x \psi^\dagger_{in}(x)\, U_{p,s}(x) \qquad d^\dagger_{in}(p,s) = \int d^3x V^\dagger_{p,s}(x)\, \psi_{in}(x) \end{aligned}$$

Reduction formula for fermions

1. remove electron from in-state

$$\langle \beta, out | \alpha; ps, in \rangle = -\frac{i}{\sqrt{z_2}} \int d^4x \langle \beta, out | \overline{\psi_\alpha}(x) | \alpha, in \rangle \overleftarrow{\left(-i\gamma^\mu \partial_\mu - m\right)}_{\alpha\beta} u(p,s)\, e^{-ip\cdot x}$$

2. remove positron(anti-particle) from in-state

$$\langle \beta, out | \alpha; \overline{ps}, in \rangle = \frac{i}{\sqrt{z_2}} \int d^4 x e^{-i\overline{p}\cdot x} \overline{v_\alpha} (\overline{p}, \overline{s}) \overrightarrow{(i\gamma^\mu \partial_\mu - m)}_{\alpha\beta} \langle \beta, out | \psi_\beta (x) | \alpha, in \rangle$$

3. remove electron from out-state

$$\langle \beta; p' s', out | \alpha, in \rangle = -\frac{i}{\sqrt{z_2}} \int d^4 x \overline{u_\alpha} (p', s') e^{ip'\cdot x} \overrightarrow{(i\gamma^\mu \partial_\mu - m)}_{\alpha\beta} \langle \beta, out | \psi_\beta (x) | \alpha, in \rangle$$

4. remove positron from out-state

$$\langle \beta; \overline{p}' \overline{s}', out | \alpha, in \rangle = \frac{i}{\sqrt{z_2}} \int d^4 x \langle \beta, out | \overline{\psi_\alpha} (x) | \alpha, in \rangle \overleftarrow{(-i\gamma^\mu \partial_\mu - m)}_{\alpha\beta} v (\overline{p},' \overline{s}') e^{-i\overline{p}'\cdot x}$$

U matrix

relation between interacting fields $\phi(x)$, $\pi(x)$ andfree fields $\phi_{in}(x)$, $\pi_{in}(x)$. Assume

$$\phi(\overrightarrow{x}, t) = U^{-1}(t)\,\phi_{in}(\overrightarrow{x}, t)\,U(t), \qquad \pi(\overrightarrow{x}, t) = U^{-1}(t)\,\pi_{in}(\overrightarrow{x}, t)\,U(t)$$

In-fields satisfy ,

$$\partial_0\phi_{in}(x) = i\left[H_{in}(\phi_{in}, \pi_{in}), \phi_{in}\right], \qquad \partial_0\pi_{in}(x) = i\left[H_{in}(\phi_{in}, \pi_{in}), \pi_{in}\right] \tag{4}$$

where $H_{in}(\phi_{in}, \pi_{in})$ is free field Hamiltonian with mass μ.
Time evolution of $\phi(x)$, $\pi(x)$ is,

$$\partial_0\phi(x) = i\left[H(\phi, \pi), \phi\right], \qquad \partial_0\pi(x) = i\left[H(\phi, \pi), \pi\right]$$

We find

$$\boxed{\mathrm{i}\frac{\partial U(t)}{\partial t} = H_I(t)\,U(t)} \tag{5}$$

Define

$$U(t, t') \equiv U(t)\,U^{-1}(t') \qquad \text{time evolution operator}$$

Eq(5) becomes,

$$i\frac{\partial U(t, t')}{\partial t} = H_I(t)\,U(t, t') \qquad \text{with} \quad U(t, t) = 1$$

convert this to integral equation

$$U(t, t') = 1 - i\int_{t'}^{t} dt_1 H_I(t_1)\,U(t_1, t')$$

which includes the initial condition. Iterate this equation assuming H_I is "small",

$$\begin{aligned} U(t,t') &= 1 - i\int_{t'}^{t} dt_1 H_I(t_1) + (-i)^2 \int_{t'}^{t} dt_1 H_I(t_1) \int_{t'}^{t_1} dt_2 H_I(t_2) + ... \\ &\quad + (-i)^n \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 ... \int_{t'}^{t_{n-1}} dt_n H_I(t_1) H_I(t_2) ... H_I(t_n) + ... \end{aligned}$$

The second term can be written as

$$\begin{aligned} U^{(2)} &= (-i)^2 \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 H_I(t_1) H_I(t_2) \\ &= (-i)^2 \int_{t'}^{t} dt_2 \int_{t_2}^{t} dt_1 H_I(t_1) H_I(t_2) \\ &= (-i)^2 \int_{t'}^{t} dt_1 \int_{t_2}^{t} dt_2 H_I(t_2) H_I(t_1) \end{aligned}$$

where we have interchange the order of integration. Renaming t_1 and t_2 ,we get

$$U^{(2)} = (-i)^2 \int_{t'}^{t} dt_1 \int_{t_2}^{t} dt_2 H_I(t_2) H_I(t_1)$$

We can use time-ordered product to combine these two equivalent expression so that the t_2 integration goes from t' to t

$$U^{(2)} = \frac{(-i)^2}{2} \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 T\left(H_I(t_2) H_I(t_1)\right)$$

We can generalize these steps to higher terms in U so that
This can be written as

$$
\begin{aligned}
U\left(t,t'\right) &= 1+\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{t'}^{t}dt_1\int_{t'}^{t}dt_2\ldots\int_{t'}^{t}dt_n\,T\left(H_I\left(t_1\right)H_I\left(t_2\right)\ldots H_I\left(t_n\right)\right)\\
&= T\left(\exp\left[-i\int_{t'}^{t}d^4x\mathcal{H}_I\left(\phi_{in},\pi_{in}\right)\right]\right)
\end{aligned}
$$

Perturbation Expansion of Vaccum expectation value

From LSZ reduction, $S - matrix$ is of the form,

$$\tau\left(x_1, x_2, ..., x_n\right) = \langle 0|T\left(\phi\left(x_1\right)\phi\left(x_2\right)...\phi\left(x_n\right)\right)|0\rangle$$

Using U matrix, write this in terms of ϕ_{in}

$$\begin{aligned}\tau &= \langle 0|T\left(U^{-1}\left(t_1\right)\phi_{in}\left(x_1\right)U\left(t_1,t_2\right)\phi_{in}\left(x_2\right)U\left(t_2,t_3\right)...U\left(t_{n-1},t_n\right)\phi_{in}\left(x_n\right)U\left(t_n\right)\right)|0\rangle \\ &= \langle 0|T\left(U^{-1}\left(t\right)U\left(t,t_1\right)\phi_{in}\left(x_1\right)...\phi_{in}\left(x_n\right)U\left(t_n,t'\right)U\left(t'\right)\right)|0\rangle\end{aligned}$$

Let $t > t_1 ... t_n > t'$,then we can pull $U^{-1}\left(t\right)$ and $U\left(t'\right)$ out of the time-ordered product, and combine $U's$ and ϕ_{in}

$$\begin{aligned}\tau &= \langle 0|U^{-1}\left(t\right)TU\left(t,t_1\right)\phi_{in}\left(x_1\right)...\phi_{in}\left(x_n\right)U\left(t_n,t'\right))U\left(t'\right)|0\rangle \\ &= \langle 0|U^{-1}\left(t\right)T(\phi_{in}\left(x_1\right)...\phi_{in}\left(x_n\right)\exp\left[-i\int_{t'}^{t}H_I\left(t"\right)dt"\right])U\left(t'\right)|0\rangle\end{aligned}$$

Theorem: $|0\rangle$ is an eigenstate of$U(-t)$ as $t \to \infty$.

Then we get the result $\langle p, \alpha, in|U(-t)|0\rangle = 0$ as $t \to \infty$ for all in-states.This means

$$U(-t)|0\rangle = \lambda_-|0\rangle \qquad \lambda_- \quad \text{some phase as } t \to \infty$$

This completes the proof.

Similarly we can show that

$$U(t)|0\rangle = \lambda_+|0\rangle \qquad \lambda_+ \quad \text{some phase as } t \to \infty$$

These phases can be written as

$$\lambda_-\lambda_+^* = [\langle 0|T\exp(-i\int_{-t}^{t} H_I\,(t')\,dt')|0\rangle]^{-1}$$

Now we have vacuum expectation value $\tau\,(x_1, x_2, ..., x_n)$ completely in terms of ϕ_{in},

$$\begin{aligned}\tau\,(x_1, x_2, ..., x_n) &= \langle 0|U^{-1}(-t)\,T\left(\phi_{in}\,(x_1)\,\phi_{in}\,(x_2)\,...\phi_{in}\,(x_n)\exp(-i\int_{-t}^{t} H_I\,(t')\,dt')\right)U(t)|0\rangle \\ &= \lambda_-\lambda_+^*\langle 0|T\left(\phi_{in}\,(x_1)\,\phi_{in}\,(x_2)\,...\phi_{in}\,(x_n)\exp(-i\int_{-t}^{t} H_I\,(t')\,dt')\right)|0\rangle\end{aligned}$$

or

$$\boxed{\tau\,(x_1, x_2, ..., x_n) = \frac{\langle 0|T\left(\phi_{in}\,(x_1)\,\phi_{in}\,(x_2)\,...\phi_{in}\,(x_n)\exp(-i\int_{-\infty}^{\infty} H_I\,(t')\,dt')\right)|0\rangle}{\langle 0|T\left(\exp(-i\int_{-\infty}^{\infty} H_I\,(t')\,dt')\right)|0\rangle}}$$

For computation we expand the exponential of H_I , to write

$$\tau(x_1, x_2, ..., x_n) = \frac{\sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_{-\infty}^{\infty} dy_1 ... dy_m \langle 0 | T(\phi_{in}(x_1) ... \phi_{in}(x_n) \mathcal{H}_I(y_1) \mathcal{H}_I(y_2) ... \mathcal{H}_I(y_m)) | 0 \rangle}{\sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_{-\infty}^{\infty} dy_1 ... dy_m \langle 0 | T(\mathcal{H}_I(y_1) \mathcal{H}_I(y_2) ... \mathcal{H}_I(y_m)) | 0 \rangle}$$

Wick's theorem

To compute product of free fields ϕ_{in} between vacuum, convert to normal ordering . Results are summarized below;

$$\begin{aligned} T(\phi_{in}(x_1) ... \phi_{in}(x_n)) \quad &= \quad : \phi_{in}(x_1) ... \phi_{in}(x_n) : \\ &+ [\langle 0 | \phi_{in}(x_1) \phi_{in}(x_2) | 0 \rangle : \phi_{in}(x_3) \phi_{in}(x_4) ... \phi_{in}(x_n) : + permutations] \end{aligned}$$

$$+ [\langle 0 | \phi_{in}(x_1) \phi_{in}(x_2) | 0 \rangle \langle 0 | \phi_{in}(x_3) \phi_{in}(x_4) | 0 \rangle : \phi_{in}(x_5) ... \phi_{in}(x_n) : + permutations] ...$$

$$+ \begin{cases} [\langle 0 | \phi_{in}(x_1) \phi_{in}(x_2) | 0 \rangle \langle 0 | \phi_{in}(x_3) \phi_{in}(x_4) | 0 \rangle ... \langle 0 | \phi_{in}(x_{n-1}) \phi_{in}(x_n) | 0 \rangle + permutations] & neven \\ [\langle 0 | \phi_{in}(x_1) \phi_{in}(x_2) | 0 \rangle ... \langle 0 | \phi_{in}(x_{n-2}) \phi_{in}(x_{n-1}) | 0 \rangle \phi_{in}(x_n) + permutations] & \text{n odd} \end{cases}$$

This can be proved by induction.
Illustrate this for n=2. Difference between $T()$ and :() :is a c-number,

$$T(\phi_{in}(x_1) \phi_{in}(x_2)) =: \phi_{in}(x_1) \phi_{in}(x_2) : + (c - number)$$

take matrix element between vacuum state,

$$\langle 0|T(\phi_{in}(x_1)\phi_{in}(x_2))|0\rangle = (c - number)$$

Then

$$T(\phi_{in}(x_1)\phi_{in}(x_2)) =: \phi_{in}(x_1)\phi_{in}(x_2): + \langle 0|T(\phi_{in}(x_1)\phi_{in}(x_2))|0\rangle$$

Most useful application of Wick's theorem

$$\begin{aligned}
\langle 0|T(\phi_{in}(x_1)...\phi_{in}(x_n))|0\rangle &= 0 \qquad \text{n odd} \\
\langle 0|T(\phi_{in}(x_1)...\phi_{in}(x_n))|0\rangle &= \sum_{permu}[\langle 0|T(\phi_{in}(x_1)\phi_{in}(x_2))|0\rangle\langle 0|T(\phi_{in}(x_3)\phi_{in}(x_4))|0\rangle...] \qquad \text{n even}
\end{aligned}$$

Notation

$$\underbrace{\phi_{in}(x_1)\phi_{in}(x_2)} = \langle 0|T(\phi_{in}(x_1)\phi_{in}(x_2))|0\rangle \qquad \text{Contraction}$$

Example:

$$\begin{aligned}
\langle 0|T(\phi_{in}(x_1)\phi_{in}(x_2) &: \phi_{in}^3(y_1) :: \phi_{in}^3(y_2) :)|0\rangle \\
&= \langle 0|T(\phi_{in}(x_1)\phi_{in}(x_2) : \phi_{in}(y_1)\phi_{in}(y_1)\phi_{in}(y_1) :: \phi_{in}(y_2)\phi_{in}(y_2)\phi_{in}(y_2) :)|0\rangle \\
&+ ...
\end{aligned}$$

Feynman Propagators

From Wick's theorem most important quantity is vacuum expecation of two free fields, called **Feynman propagator**.

$$\begin{aligned}\langle 0| T(\phi_{in}(x)\phi_{in}(y))|0\rangle &= i\triangle_F\left(x-y,\mu^2\right) = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2-\mu^2+i\varepsilon} \\ &= i\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} i\triangle_F(k) \\ \text{with}\quad i\triangle_F(k) &= \frac{i}{k^2-\mu^2+i\varepsilon}\end{aligned}$$

For complex scalar field

$$\langle 0| T(\phi_{in}(x)\phi^*_{in}(y))|0\rangle = i\triangle_F\left(x-y,\mu^2\right) = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2-\mu^2+i\varepsilon}$$

Fermion field

$$\begin{aligned}\langle 0| T(\psi^{in}_\alpha(x)\overline{\psi}^{in}_\beta(y))|0\rangle &= iS_F(x-y,m)_{\alpha\beta} \\ &= i\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)}\frac{\left(\gamma^\mu p_\mu + m\right)_{\alpha\beta}}{p^2-m^2+i\varepsilon} = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} iS_F(p)_{\alpha\beta}\end{aligned}$$

photon field

$$\langle 0| T(A_\mu^{in}(x) A_\upsilon^{in}(y))|0\rangle = iD_F^{tr}(x-y) = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2+i\varepsilon} \times \left[-g_{\mu\nu} - \frac{k_\mu k_\nu}{(k\cdot\eta)^2-k^2} + \frac{(k\cdot\eta)\left(k_\mu\eta_\nu + k_\nu\eta_\mu\right)}{(k\cdot\eta)^2-k^2} - \frac{k^2\eta_\mu\eta_\nu}{(k\cdot\eta)^2-k^2}\right]$$

where $\eta_\mu = (1,0,0,0)$
It can be shown that in QED only term contributes is "$-g_{\mu\nu}$" as a consequence of the gauge invariance.

Graphical representation
Each line (propagator) represents a contraction in Wick's expansion
e.q.

$$\underset{y}{\circ}\cdots\cdots\underset{x}{\circ} \qquad i\triangle_F\left(x-y,\mu^2\right)$$

$$^{\beta}_{y}\text{---}>\text{---}^{\alpha}_{x} \qquad iS_F\left(x-y,m\right)_{\alpha\beta}$$

$$\sim^{\nu}\sim\sim\sim\sim\sim\sim^{\mu} \qquad iD_F^{tr}\left(x-y\right)$$

Vaccum Amplitude
In the denominator of $\tau-$function, there are no external lines

$$\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int_{-\infty}^{\infty}d^4y_1...d^4y_m\langle 0|T\left(\mathcal{H}_I\left(\phi_{in}\left(y_1\right)\right)...\mathcal{H}_I\left(\phi_{in}\left(y_m\right)\right)\right)|0\rangle$$

e.q. 2nd order term for the case $\mathcal{H}_I=\frac{\lambda}{3!}:\phi_{in}^3:$

$$
\begin{aligned}
\langle 0|T\left(\mathcal{H}_I\left(\phi_{in}\left(y_1\right)\right)\mathcal{H}_I\left(\phi_{in}\left(y_m\right)\right)\right)|0\rangle &= \left(\frac{\lambda}{3!}\right)^2 \langle 0|T\left(:\phi_{in}^3\left(y_1\right)::\phi_{in}^3\left(y_2\right):\right)|0\rangle \\
&= \left(\frac{\lambda}{3!}\right)^2 :\phi_{in}\left(y_1\right)\phi_{in}\left(y_1\right)\phi_{in}\left(y_1\right)::\phi_{in}\left(y_2\right)\phi_{in}\left(y_2\right)\phi_{in}\left(y_2\right): 3\times 2
\end{aligned}
$$

<u>closed loop diagram</u> :graphs with no external lines(lines with open end)
<u>disconnected diagram</u> :a subgraph not connected to any external lines
<u>connected diagram</u> :graph not disconnected

All graphs appearing in the numerator of the $\tau-$function can be seperated uniquely into connected and disconnected parts. It turns out that disconnected part is cancelle by those in denominator.
Example :$\mathcal{H}_I = \frac{\lambda}{3!}\phi_{in}^3$

$$\phi(q_1) + \phi(q_2) \longrightarrow \phi(p_1) + \phi(p_2)$$

$$\begin{aligned} S_{\beta\alpha} &= \langle \beta, out | \alpha, in \rangle = \langle p_1, p_2, out | q_1, q_2, in \rangle \\ &= \left(\frac{-i}{\sqrt{z}}\right)^4 \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 e^{ip_1y_1} e^{ip_2y_2} \left(\Box_{y_1} + \mu^2\right)\left(\Box_{y_2} + \mu^2\right) \langle 0 | T\left(\phi(y_1)\phi(y_2)\phi(x_1)\phi(x_2)\right) |0 > \\ &\quad \left(\overleftarrow{\Box}_{x_1} + \mu^2\right)\left(\overleftarrow{\Box}_{x2} + \mu^2\right) e^{-iq_1x_1} e^{-iq_2x_2} \\ &= \left(\frac{-i}{\sqrt{z}}\right)^4 \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \left(\mu^2 - p_1^2\right)\left(\mu^2 - p_2^2\right)\left(\mu^2 - q_1^2\right)\left(\mu^2 - q_2^2\right) \\ &\quad \times \tau(y_1, y_2, x_1, x_2) e^{i(p_1y_1 + p_2y_2)} e^{-i(q_1x_1 + q_2x_2)} \end{aligned}$$

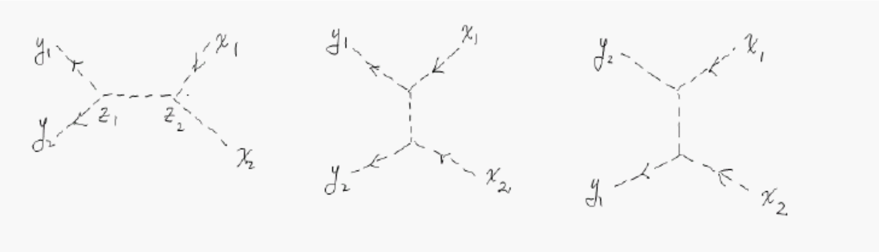
Perturbation expansion of $\tau-$function

$$\tau(y_1, y_2, x_1, x_2) = \sum_n \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d^4z_1 ... d^4z_n \langle 0 | T\left(\phi_{in}(y_1)\phi_{in}(y_2)\phi_{in}(x_1)\phi_{in}(x_2)\mathcal{H}_I(\phi_{in}(z_1)) ... \mathcal{H}_I(\phi_{in}(z_n))\right) |0\rangle$$

Lowest order contribution

$$\tau^{(2)}(y_1, y_2, x_1, x_2) = \frac{(-i)^2}{2!} \int_{-\infty}^{\infty} d^4z_1 d^4z_2 \langle 0 | T\left(\phi_{in}(y_1)\phi_{in}(y_2)\phi_{in}(x_1)\phi_{in}(x_2)\left(\frac{\lambda}{3!}\phi_{in}^3(z_1)\right)\left(\frac{\lambda}{3!}\phi_{in}^3(z_2)\right)\right) |0\rangle$$

Using Wick's theorem, the connected diagrams are,



Their contribution to $\tau(y_1, y_2, x_1, x_2)$ is

$$\tau^{(2)}(y_1, y_2, x_1, x_2) = \frac{(-i\lambda)^2}{2!} \int_{-\infty}^{\infty} d^4 z_1 d^4 z_2 \, i\triangle_F(y_1 - z_1)\, i\triangle_F(y_2 - z_1)$$
$$i\triangle_F(z_2 - x_1)\, i\triangle_F(z_2 - x_2)\, i\triangle_F(z_1 - z_2) + \ldots$$

use the propagator in momentum space

$$i\triangle_F(x) = \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\varepsilon} e^{-ik\cdot x}$$

Then

$$\tau^{(2)}(y_1,y_2,x_1,x_2) = \frac{(-i\lambda)^2}{2!}\int_{-\infty}^{\infty} d^4z_1 d^4z_2 \int \frac{d^4k_1}{(2\pi)^4}\int \frac{d^4k_2}{(2\pi)^4}\cdots\int \frac{d^4k_5}{(2\pi)^4}$$
$$e^{-ik_1\cdot(y_1-z_1)} i\triangle_F(k_1)\, e^{-ik_2\cdot(z_1-x_1)} i\triangle_F(k_2)$$
$$e^{-ik_3\cdot(z_1-z_2)} i\triangle_F(k_3)\, e^{-ik_4\cdot(y_2-z_2)} i\triangle_F(k_4)\, e^{-ik_5\cdot(z_2-x_2)} i\triangle_F(k_5)$$

$$z_1 \text{ integration} \quad \int d^4z_1 e^{i(k_1-k_2-k_3)\cdot z_1} = (2\pi)^4\delta^4(k_1-k_2-k_3)$$
$$z_2 \text{ integration} \quad \int d^4z_2 e^{i(k_3+k_4-k_5)\cdot z_2} = (2\pi)^4\delta^4(k_3+k_4-k_5)$$

energy-momentum conservation at each vertex
Then

$$\tau^{(2)}(y_1,y_2,x_1,x_2) = \frac{(-i\lambda)^2}{2!}\int \frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_4}{(2\pi)^4}(2\pi)^4\delta^4(k_1-k_2+k_4-k_5)$$
$$i\triangle_F(k_1)\, i\triangle_F(k_2)\, i\triangle_F(k_4)\, i\triangle_F(k_5)\, i\triangle_F(k_1-k_2)\, e^{-ik_1\cdot y_1} e^{ik_2\cdot x_1} e^{-ik_4\cdot y_2} e^{ik_5\cdot x_2}$$

$$\int \tau^{(2)}(y_1,y_2,x_1,x_2)\, d^4x_1 d^4x_2 d^4y_1 d^4y_2 e^{i(p_1y_1+p_2y_2)} e^{-i(q_1x_1+q_2x_2)} e^{-ik_1\cdot y_1} e^{ik_2\cdot x_1} e^{-ik_4\cdot y_2} e^{ik_5\cdot x_2}$$
$$= (2\pi)^4\delta^4(k_2-q_1)(2\pi)^4\delta^4(k_5-q_2)(2\pi)^4\delta^4(p_1-k)(2\pi)^4\delta^4(p_2-k_4)$$

We see that the external line propagators cancell out and

$$S_{\beta\alpha} = \frac{(-i\lambda)^2}{2!} \left(\frac{1}{\sqrt{z}} \right)^4 [\frac{i}{(q_1 + q_2) - \mu^2}] (2\pi)^4 \delta^4 (p_1 + p_2 - q_1 - q_2) + ...$$

This is rather simple answer in momentum space.

Cross section and Decay rate

Write the S-matrix elements as

$$S_{fi} = \delta_{fi} + i(2\pi)^4\delta^4(p_f - p_i)T_{fi}$$

T_{fi} : invariant amplitude for $i \longrightarrow f$.
For $i \neq f$, the transistion probability is

$$|S_{fi}|^2 = (2\pi)^4\delta^4(0)[(2\pi)^4\delta^4(p_f - p_i)\,|T_{fi}|^2]$$

To interprete $\delta^4(0)$, we write

$$(2\pi)^4\delta^4(p_f - p_i) = \int d^4x e^{-i(p_f - p_i)x}$$

The integration is over some large but finite volume V and time interval T.
Then we can interprete $\delta^4(0)$ as

$$(2\pi)^4\delta^4(0) = VT$$

and write

$$|S_{fi}|^2 = VT[(2\pi)^4\delta^4(p_f - p_i)\,|T_{fi}|^2$$

The transistion rate (transistion probability per unit time) is then

$$\omega_{fi} = (2\pi)^4\delta^4(p_f - p_i)\,|T_{fi}|^2\, V$$

Decay rates

For a general decay processes with kinematics,

$$a(p) \rightarrow c_1(k_1) + c_2(k_2) + ... + c_n(k_n) \quad p_f = \sum_{l=1}^{n} k_i \quad p_i = p$$

number of states in the volume elements $d^3k_1 \, ...d^3k_n$ in momentum space is

$$\prod_{l=1}^{n} \frac{d^3k_l}{(2\pi)^3 2\omega_{kl}}$$

The transition rate, summing over final states is

$$d\omega' = (2\pi)^4 \delta^4(p - \Sigma_{j=1}^{n} k_j) \, |T_{fi}|^2 \, V \prod_{l=1}^{n} \frac{d^3k_l}{(2\pi)^3 2\omega_{kl}}$$

For the invariant normalization of the physical states

$$< p|p' >= (2\pi)^3 \delta^3(\vec{p} - \vec{p}\,')2\omega_p$$

For $p = p'$,

$$< p|p >= (2\pi)^3 \delta^3(0)2\omega_p = 2V\omega_p$$

which is the number of particle in the initial state.
The decay rate per particle is then

$$d\omega = \frac{d\omega'}{2V\omega_p} = (2\pi)^4 \delta^4(p - \Sigma_{j=1}^{n} k_j) \, |T_{fi}|^2 \, \frac{1}{2\omega_p} \prod_{l=1}^{n} \frac{d^3k_l}{(2\pi)^3 2\omega_{kl}}$$

If there are "m" identical particles in the final state, divide this by m!

$$d\omega = \frac{1}{2\omega_p} \left|T_{fi}\right|^2 \frac{d^3 k_1}{(2\pi)^3 2\omega_1} \cdots \frac{d^3 k_n}{(2\pi)^3 2\omega_n} (2\pi)^4 \delta^4 (p - \Sigma_{j=1}^{n} k_j) S \qquad S = \prod_j \frac{1}{(m_j)!}$$

Cross section

For a scattering processes ,

$$a(p_1) + b(p_2) \longrightarrow c_1(k_1) + c_2(k_2) + ... + c_n(k_n)$$

the transition rate is, after summing over final states,

$$d\omega' = (2\pi)^4\delta^4(p_1 + p_2 - \Sigma_{j=1}^n k_j)\,|T_{fi}|^2\, V \prod_{l=1}^{n} \frac{d^3k_l}{(2\pi)^3 2\omega_{kl}}$$

Normalize this to 1 particle in the beam and 1 particle in the target and divide this by the flux∼relative velocity divided by the volume, to get differential cross section

$$d\sigma = \frac{1}{2\omega_{p_1}V}\frac{1}{2\omega_{p_2}V}(2\pi)^4\delta^4(p_1 + p_2 - \Sigma_{j=1}^n k_j)\,|T_{fi}|^2\, V \prod_{l=1}^{n} \frac{d^3k_l}{(2\pi)^3 2\omega_{kl}}\frac{V}{|\vec{v_1} - \vec{v_2}|}$$

Velocity factor can be written as

$$I = |\vec{v_1} - \vec{v_2}| = \left|\frac{\vec{p_1}}{E_1} - \frac{\vec{p_2}}{E_2}\right|$$

In the C.M. frame $\vec{p_1} = -\vec{p_2} = \vec{p} \qquad p_1 = (E_1, \vec{p}), p_2 = (E_2, -\vec{p})$

$$I = \frac{|\vec{p}|}{E_1E_2}(E_1 + E_2)$$

$$(p_1 \cdot p_2)^2 = (E_1E_2 + \vec{p}^{\,2})^2 = E_1^2E_2^2 + 2E_1E_2\vec{p}^{\,2} + \vec{p}^{\,4}$$

$$
\begin{aligned}
(p_1 \cdot p_2)^2 - m_1^2 m_2^2 &= (\vec{p}^{\,2} + m_1^2)(\vec{p}^{\,2} + m_2^2) + 2E_1 E_2 \vec{p}^{\,2} + \vec{p}^{\,4} - m_1^2 m_2^2 \\
&= \vec{p}^{\,2}[2\vec{p}^{\,2} + (m_1^2 + m_2^2) + 2E_1 E_2] \\
&= \vec{p}^{\,2}(E_1 + E_2)^2
\end{aligned}
$$

$$
\Rightarrow I = \frac{1}{E_1 E_2}\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}
$$

$$
d\sigma = \frac{1}{I}\frac{1}{2\omega_{p_1}}\frac{1}{2\omega_{p_2}}(2\pi)^4 \delta^4(p_1 + p_2 - \Sigma_{j=1}^{n} k_j)\,|T_{fi}|^2 \prod_{l=1}^{n} \frac{d^3 k_l}{(2\pi)^3 2\omega_l}
$$

Feynman Rules

Since final forms for T_{fi} are quite simple, can use simple rules to sidestep all those tedious intermediate steps.
Draw all connected Feynman graphs with appropriate external lines.
Label each with momenta and impose momentum conservation for each vertex.

1. For each internal fermion line with momentum p,enter the propagator

$$iS_F(p) = \frac{i}{p^\mu \gamma_\mu - m + i\varepsilon}$$

2. For each internal boson line of spin 0,with momentum q,enter the propagator

$$i \triangle_F(q) = \frac{i}{q^2 - \mu^2 + i\varepsilon}$$

3. For each internal photon line with momentum k,enter the propagator

$$iD_F(k)_{\mu v} = \frac{-ig_{\mu v}}{k^2 + i\varepsilon}$$

4. For each internal momentum l not fixed by momentum conservation,enter

$$\int \frac{d^4 l}{(2\pi)^4}$$

5. For each closed fermion loop,enter (-1) . Also a factor of (-1) between graphs which differ by an interchange of two external identical fermion lines.

6. At each vertex, the factors depend on form of interactions.

a. $\frac{1}{3!}\ \lambda\phi^3 \qquad (-i\lambda)$

b. $\frac{1}{4!}\ \lambda\phi^4 \qquad (-i\lambda)$

c. $e\overline{\psi}\gamma_\mu\psi A^\mu \qquad \left(-ie\gamma_\mu\right)$

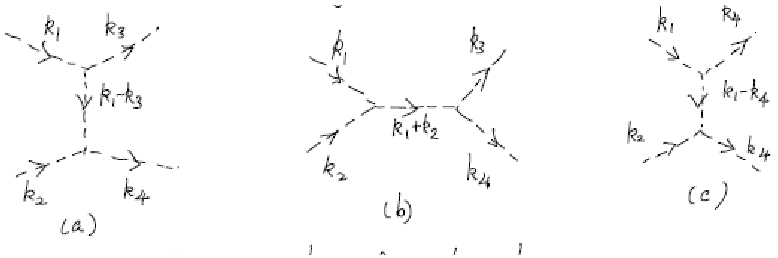
d. $f\ \overline{\psi}\gamma_5\psi\phi \qquad (-if\gamma_5)$

Example in $\lambda\phi^3$ theory

In $\lambda\phi^3$ theory, consider the processes $\phi(k_1) + \phi(k_2) \longrightarrow \phi(k_3) + \phi(k_4)$

To order λ^2 , we have 3 Feynman diagrams

We can write down the matrix element for each graph,



$$T^{(a)} = (-i\lambda)^2 \frac{i}{(k_1 - k_3)^2 - \mu^2} \qquad T^{(b)} = (-i\lambda)^2 \frac{i}{(k_1 + k_2)^2 - \mu^2} \qquad T^{(c)} = (-i\lambda)^2 \frac{i}{(k_1 - k_4)^2 - \mu^2}$$

Total amplitude $T = T^{(a)} + T^{(b)} + T^{(c)}$
Mandelstam variables

$$\begin{aligned} s &= (k_1 + k_2)^2 && \text{total energy in c.m. frame} \\ t &= (k_1 - k_3)^2 && \text{momentum transfer(scattering angle)} \\ u &= (k_1 - k_4)^2 \end{aligned}$$

$$s + t + u = 4\mu^2$$

Usually these amplitudes are written as

$$T^{(a)} = (\lambda)^2 \frac{i}{t - \mu^2} \qquad T^{(b)} = (\lambda)^2 \frac{i}{s - \mu^2} \qquad T^{(c)} = (\lambda)^2 \frac{i}{u - \mu^2}$$