Quantum Field Theory

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Homework set 5, Due Wed Dec 24

1. Consider the reaction

$$e^+(p') + e^-(p) \to \mu^+(k') + \mu^-(k)$$

   (a) The spin averaged probability is of the form

$$\frac{1}{4}\sum_{spin}\left|M(e^+e^- \to \mu^+\mu^-)\right|^2 = \frac{e^4}{q^4} Tr\left[\left(\not{p}' - m_e\right)\gamma^\mu\left(\not{p} + m_e\right)\gamma^\nu\right] Tr\left[\left(\not{k}' + m_\mu\right)\gamma_\mu\left(\not{k} + m_\mu\right)\gamma^\nu\right]$$

   Show that for energies $\gg m_\mu$, this can be written as

$$\frac{1}{4}\sum_{spin'}\left|M(e^+e^- \to \mu^+\mu^-)\right|^2 = 8\frac{e^4}{q^4}\left[(p\cdot k)\left(p'\cdot k'\right) + (p'\cdot k)\left(p\cdot k'\right)\right]$$

   (b) The phase space for this reaction is given by

$$\rho = \int (2\pi)^4\delta^4(p + p' - k - k')\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}$$

   Show that

$$\rho = \frac{d\Omega}{32\pi^2}$$

   in the center of mass frame.

2. The Lagrangian for the free photon is of the form,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

   Suppose we add a mass term to this Lagrangian

$$\mathcal{L}' = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\mu^2 A_\mu A^\mu$$

   (a) Find the equation of motion.

   (b) From the equation of motion show that

$$\partial^\mu A_\mu = 0$$

   and use the equation of motion to express A_0 in terms of other field variables

   (c) Show that the polarization vectors $\epsilon(\vec{k},\lambda)$, $\lambda = 1, 2, 3$ can be chosen so that

$$\sum_\lambda \epsilon^\mu(\vec{k},\lambda)\epsilon^\nu(\vec{k},\lambda) = g^{\mu\nu} - \frac{k^\mu k^\nu}{\mu^2}$$

3. In the $\lambda\phi^4$ theory the interacting Lagrangian is of the form,

$$\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$$

   For the 2-body elastic scattering we need to compute to second order in λ the following vacuum expectation value

$$\tau^{(2)}(y_1, y_2, x_1, x_2) = \frac{(-i)^2}{2!}\int_{-\infty}^{\infty} d^4z_1 d^4z_2 \langle 0|T\left(\phi_{in}(y_1)\phi_{in}(y_2)\phi_{in}(x_1)\phi_{in}(x_2)\left(\frac{\lambda}{4!}\phi_{in}^4(z_1)\right)\left(\frac{\lambda}{4!}\phi_{in}^4(z_2)\right)\right)|0\rangle$$

   Use Wick's theorem to write this matrix element in terms of propagators.

4. The Lagrangian for the free fermion field is of the form,

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

Compute the free propagator

$$\int d^4x e^{ipx} \left\langle 0 \middle| T\left(\psi_\alpha\left(x\right)\bar{\psi}_\beta\left(0\right)\right) \middle| 0 \right\rangle$$