Quantum Field Theory
Chap 7 Renormalization

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Contents

1 Renormalization

Many people who have studied relativistic quantum field theory find the most difficult part is the theory of renormalization. The relativistic field theory is full of infinities which need to be taken care of before the theoretical predictions can be compared with experimental measurements. These infinities look formidable at first sight. It is remarkable that over the years *consistent* ways have been found to make sense of these apparently divergent theories and the results compared successfully with experiments.

The theory of renormalization is a prescription which consistently isolates and removes all these infinities from the physically measurable quantities. Note that the need for renomalization is quite general and is not unique to the relativistic field theory. For example, consider an electron moving inside a solid. If the interaction between electron and the lattice of the solid is weak enough, we can use an effective mass m^* to describe its response to an externally applied force and this effective mass is certainly different from the mass m measured outside the solid where there is no interaction. Thus the electron mass is changed (renormalized) from m to m^* by the interaction of the electron with the lattice in the solid. In this simple case, both m and m^* are measurable and hence *both are* finite. For the relativistic field theory, the situation is the same except for two important differences. First the renormalization due to the interaction is generally infinite (corresponding to the divergent loop diagrams). These infinities, coming from the contribution of high momentum modes are present even for the cases where the interactions are weak. Second, there is no way to switch off the interaction between particles and the quantities in the absence of interaction, bare quantities, are not measurable. Roughly speaking, the program of removing the infinities from physically measurable quantities in relativistic field theory, the renormalization program, involves shuffling all the divergences into bare quantities which are not measurable. In other words, we can redefine the unmeasurable quantities to absorb the divergences so that the physically measurable quantities are finite. The renormalized mass which is now finite can only be determined from experimental measurement and can not be predicted from the theory alone.

Eventhough the concept of renormalization is quite simple, the actual procedure for carrying out the operation is quite complicated and intimidating. In this chapter, we will give a bare bone of this program. Note that we need to use some regularization procedure ([**?**]) to make these divergent quantities finite before we can do mathematically meaningful manipulations. It turns out that not every relativistic field theory will have this property that all divergences can be absorbed into redefinition of few physical parameters. Those which have this property are called *renormalizable* theories and those which don't are called unrenormalizable theories. This has became an important criteria for choosing a right theory because we do not really know how to handle the *unrenormalizable* theory.

1.1 Renormalization in $\lambda\phi^4$ Theory

To avoid complicaiton from spin and gauge dependence, we consider a simple example of $\lambda\phi^4$ theory where the Lagrangian is given by,

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

$$\mathcal{L}_0 = \frac{1}{2}[(\partial_\mu\phi_0)^2 - \mu_0^2\phi_0^2] \quad , \quad \mathcal{L}_I = -\frac{\lambda_0}{4!}\phi_0^4$$

♠Here μ_0^2 , λ_0, ϕ_0 are usually called the bare or unrenormalized quantities and will be renormalized by the interactions.♠

Feynman rule

The vertex and propagator are given by

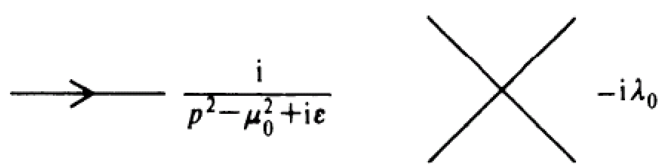


Fig 1 Feynman rule for $\lambda\varphi^4$ theory

Here p is the momentum carried by the line and μ_0^2 is the bare mass term in $\mathcal{L}_0$.

(a) 4-momentum conservation at each vertex.

(b) Integrate over internal momenta not fixed by momentum conservation

(c) For the discussion of renormalization, we do not give propagators for external lines

Simple example

♠First consider♠ the two point function (propagator) is defined by

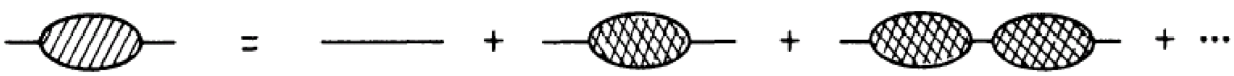
$$i\Delta(p) = \int d^4x e^{-ip\cdot x} \langle 0 | T(\varphi_0(x)\varphi_0(0)) | 0 \rangle$$

and has contribution from following graphs for example,

(a) (b)

We define the term **one-particle-irreducible**, or 1PI as those graphs which can not be made disconnected by cutting any one internal line. It is clear that we can express a general graph in terms of some collections of 1PI graphs connected by a single line. For expample in the above graphs diagram (a) is an 1PI graph while diagram (b) is not. It is not hard to see that we can attribute all the divergence to the 1PI subgraphs. For example, in daigram (b) the divergences are due to the 1PI in 1-loop. So we can concentrate on 1PI graphs for the discussion of renormalization ♠because in the one particle reducible graph the internal line which can be cut to make the graphs a disconnected one will not be integrated over.♠

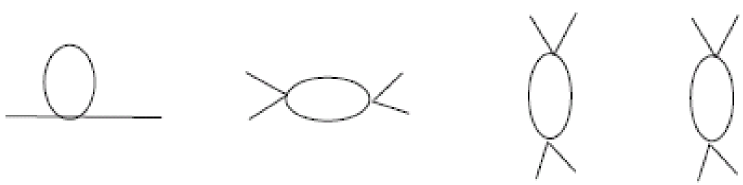
It is easy to see that we can write the full propagator in terms of 1PI self energy graph as a geometric series,



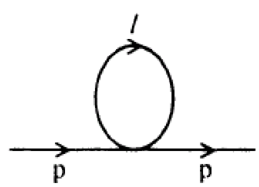
$$\begin{aligned} i\Delta(p) &= \frac{i}{p^2-\mu_0^2+i\varepsilon}+\frac{i}{p^2-\mu_0^2+i\varepsilon}\left(-i\Sigma\left(p^2\right)\right)\frac{i}{p^2-\mu_0^2+i\varepsilon}+\cdots \\ &= \frac{i}{p^2-\mu_0^2-\Sigma\left(p^2\right)+i\varepsilon} \end{aligned} \tag{1}$$

Here $\Sigma\left(p^2\right)$ contains all IPI self energy graphs.

In one-loop we have following divergent ♠1PI♠ graphs,

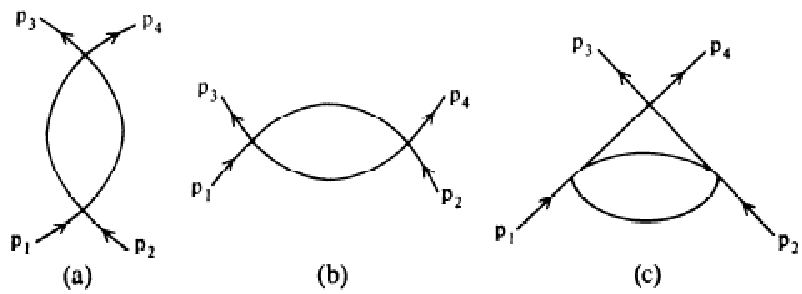


The 1-loop self energy graph given below,



$$-i\Sigma(p)=-\frac{i\lambda_0}{2}\int\frac{d^4l}{(2\pi)^4}\frac{i}{l^2-\mu_0^2+i\varepsilon}$$

is quadratically divergent. The 1-loop 4-point function are



with graph (a) gives the contribution

$$\Gamma(p^2) = \frac{(i\lambda_0)^2}{2}\int \frac{d^4l}{(2\pi)^4}\frac{i}{(l-p)^2-\mu_0^2+i\varepsilon}\frac{i}{l^2-\mu_0^2+i\varepsilon}$$

and is logarithmically divergent. Note that in $\Gamma\left(p^2\right)$, inside the integral the dependence on the external momentum p is in the combination $(p-l)$ in the denominator. This means that if we differentiate $\Gamma\left(p^2\right)$ with respect to the external momenta p, power of l will increase in denominator and will make the integral more convergent,

$$\frac{\partial}{\partial p^2}\Gamma(p^2) = \frac{1}{2p^2}p_\mu\frac{\partial}{\partial p_\mu}\Gamma(p^2) = \frac{\lambda_0^2}{p^2}\int\frac{d^4l}{(2\pi)^4}\frac{(l-p)\cdot p}{[(l-p)^2-\mu_0^2+i\varepsilon]^2}\frac{1}{l^2-\mu_0^2+i\varepsilon} \longrightarrow \quad \text{convergent}$$

This is true whenever the external momenta ocurrs in company with internal momenta in the denominator. Thus if we expand $\Gamma(p^2)$ in Taylor series,

$$\Gamma(p^2) = a_0 + a_1p^2 + \ldots$$

the coefficients are all derivatives of $\Gamma(p^2)$ with respect to p and the divergences are contained in first few terms in the expansion. So we will use Taylor expansion to separate the divergent quantities from the convergent terms. In our simple case, if we write

$$\Gamma(p^2) = \Gamma(0) + \tilde{\Gamma}(p^2)$$

then then only the first term $\Gamma(0)$ is divergent and second term $\tilde{\Gamma}(p^2)$ which contains all higher derivatives terms is finite. Other 1-loop graphs are either finite or contain the above graphs as subgraphs.

1.1.1 Mass and wavefunction renormalization

In 1PI self energy, the expansion in external momentum p will have 2 divergent terms because it is quadratically divergent,

$$\Sigma(p^2) = \Sigma(\mu^2) + (p^2-\mu^2)\Sigma'^2(\mu^2) + \tilde{\Sigma}(p^2) \tag{2}$$

where μ^2 is an arbitrary reference point for the Taylor expansion. It is clear that the first term $\Sigma(\mu^2)$ is quadratically and the second term $\Sigma'^2(\mu^2)$ logarithmically divergnet. The 3rd term $\tilde{\Sigma}(p^2)$ is finite and satisfies the conditions,

$$\tilde{\Sigma}(\mu^2) = 0, \qquad \tilde{\Sigma}'^2(\mu^2) = 0$$

Complete propagator is then

$$i\Delta(p^2) = \frac{i}{p^2 - \mu_0^2 - \Sigma(\mu^2) - (p^2 - \mu^2)\Sigma'^2 - \tilde{\Sigma}(p^2)}$$

Suppose we choose μ^2 such that

$$\mu_0^2 - \Sigma(\mu^2) = \mu^2 \quad \text{mass renormalization} \tag{3}$$

then $\Delta(p^2)$ will have a pole at $p^2 = \mu^2$. The position of the pole μ^2 can be interpreted as physical mass and μ_0^2 is the bare mass which appears in the original Lagrangian. Note that in Eq *(*3) $\Sigma(\mu^2)$ is divergent. Thus we require the bare mass μ_0^2 to be divergent in such a way that the combination $\mu_0^2 - \Sigma(\mu^2)$, the physical mass, is finite.♠ In other words, we blame the divergence on the bare mass μ_0^2, which is not a measurable quantity and diverges in such a way that the combination with the term from the interaction $\Sigma(\mu^2)$ is finite. This interpretation looks strange and is hard to accept at first sight but it is logically consistent and workable♠

Using the mass renormalization condition in Eq (3) we can write the full propagator as

$$i\Delta(p^2) = \frac{i}{(p^2 - \mu^2)[1 - \Sigma'^2\left(\mu^2\right)] - \tilde{\Sigma}(p^2)}$$

Since $\Sigma'(\mu^2)$ and $\tilde{\Sigma}(p^2)$ are both of order λ_0 or higher, we can approximate

$$\tilde{\Sigma}(p^2) \to (1 - \Sigma'^2\left(\mu^2\right))\tilde{\Sigma}(p^2) + O\left(\lambda_0^2\right)$$

♠ Note that the term we add $-\Sigma'^2\left(\mu^2\right)\tilde{\Sigma}(p^2)$ is of order λ_0^2 and is negligible in the perturbative expansion in λ_0. The purpose for doing this is to be able to factor out the combination $(1 - \Sigma'^2\left(\mu^2\right))$ to wrtie the full propagator as♠

$$i\Delta(p^2) = \frac{iZ_\phi}{p^2 - \mu^2 - \tilde{\Sigma}(p^2) + i\varepsilon} \quad \text{with} \quad Z_\phi = \frac{1}{1 - \Sigma'^2(\mu^2)} \approx 1 + \Sigma'^2(\mu^2)$$

where we have expand the denominator as

$$\frac{1}{1 - \Sigma'^2(\mu^2)} \approx 1 + \Sigma'^2(\mu^2) + O\left(\lambda_0^2\right)$$

Now the divergence is aggravited into a multiplicative factor Z_ϕ and can be removed by rescaling the field,

$$\phi = \frac{1}{\sqrt{Z_\phi}}\phi_0$$

so that the propagator for ϕ is

$$i\Delta_R(p) = \int d^4x e^{-ipx} \ \langle 0|T(\phi(x)\phi(0))|0\rangle = \frac{1}{Z_\phi} i\Delta(p^2) = \frac{i}{p^2 - \mu^2 - \tilde{\Sigma}(p^2) + i\varepsilon}$$

which is now completely finite. Z_ϕ is usually called the **wave function renormalization constant**. The new field ϕ is called the renormalized field operator. ♠ Again we shuffle the divergence into the bare or unrenormalized field ϕ_0 so that the renormalized field ϕ will have finite matrix elements.♠

Note that this method of removing the first few terms in the Taylor expansion about certain point is sometime called the **substration scheme**. In our case we expand the self energy $\Sigma(p^2)$ around the point μ^2 as in Eq (2) and subtract the first two terms in the expansion, $\Sigma^2(\mu^2)$, and $\left(p^2 - \mu^2\right)\Sigma'^2(\mu^2)$. The left over $\tilde{\Sigma}(p^2)$ will satifies the conditions

$$\tilde{\Sigma}(p^2)\Big|_{p^2=\mu^2} = 0, \qquad \frac{\partial\tilde{\Sigma}(p^2)}{\partial p^2}\Bigg|_{p^2=\mu^2} = 0 \tag{4}$$

In fact, the conditions in Eq (4) contain all the information about the expansion points and terms to be subtracted. Very often these conditions sometimes are used to specify the normalization procedure.

Note also that since our purpose is to remove the divergence we do not have to choose the expansion point μ^2 to be the physical mass as we have done. Suppose we use some other point μ_1^2 for the Taylor expansion,

$$\Sigma(p^2) = \Sigma(\mu_1^2) + (p^2 - \mu_1^2)\Sigma'^2(\mu_1^2) + \tilde{\Sigma}(p^2)$$

Then the full propagator is

$$i\Delta(p^2) = \frac{i}{p^2 - \mu_0^2 - \Sigma(\mu_1^2) - (p^2 - \mu_1^2)\Sigma'^2(\mu_1^2) - \tilde{\Sigma}(p^2)}$$

We can still define the physical mass μ^2 to be the value of p^2 where the propagator $i\Delta(p^2)$ has a pole, i.e.

$$\mu^2 - \mu_0^2 - \Sigma(\mu_1^2) - (\mu^2 - \mu_1^2)\Sigma'^2(\mu_1^2) - \tilde{\Sigma}(\mu^2) = 0$$

This will give the physical mass μ^2 as a complicate function of bare mass μ_0^2 divergent quantities, ♠$\Sigma(\mu_1^2), \Sigma'^2(\mu_1^2)$ and convergent one $\tilde{\Sigma}(\mu^2)$. The parameter μ_1^2 is now a parameter of the theory and all the physical quantities are then functions of this parameter. Its value needs to be determined by comparing the theoretical calculation of certain physical quantity with the experimental measurement. This might looks cubersome, but is workable in principle.♠

For more general Green's functions of the renomalized fields, we define the renormalized Green's functions in terms of renormalized fields,

$$G_R^{(n)}(x_1 \ldots x_n) = \langle 0|T(\phi(x_1) \cdots \phi(x_n))|0\rangle$$

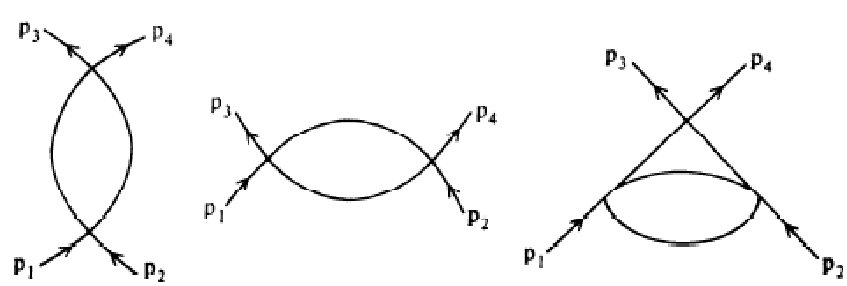
$$= Z_\phi^{-n/2} \langle 0|T(\phi_0(x_1) \cdots \phi_0(x_n))|0\rangle = Z_\phi^{-n/2} G_0^{(n)}(x_1 \ldots x_n)$$

♠so that the renormalized Green's function will be free of divergences.♠

In summary, the mass and wave function renormalization remove all the divergence in 1PI self energy graphs by redefining the mass and field operator. In other words, all the divergences have been shuffled into unobserable bare mass μ_0^2 and bare field operator ϕ_0. The program of renormalization in $\lambda\phi^4$ theory is to show that this procedure can be extended to higher order consistently. This is a daunting task and will not be discussed here. Later we will illustrate with counter terms scheme to make it more plausible.

1.1.2 Coupling constant renormalization

For 1PI 4-point functions $\Gamma^{(4)}(p_1 \cdots p_4)$, there are 3 1-loop diagrams,



(a) (b) (c)

If we include the lowest order tree diagram, we have

$$\Gamma_0^{(4)}(s,t,u) = -i\lambda_0 + \Gamma(s) + \Gamma(t) + \Gamma(u)$$

where

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2, \qquad s + t + u = 4\mu^2$$

are the standard Mandelstam variables. Since these contributions are logarithmically divergent, we need to isolate only one term in the Taylor expansion, or one substraction, to make this finite. To remove this divergence from the 4-point function, we need to make the substraction of 4-point function at some kinematical point. For conveience we can choose the symmetric point $s_0 = t_0 = u_0 = \dfrac{4\mu^2}{3}$,

$$\Gamma_0^{(4)}(s,t,u) = -i\lambda_0 + 3\Gamma(s_0) + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)$$

where

$$\tilde{\Gamma}(s) = \Gamma(s) - \Gamma(s_0),$$

is finite. We now combine the divergent terms and define the coupling constant renormalization constant Z_λ by

$$-i\lambda_0 + 3\Gamma(s_0) = -iZ_\lambda^{-1}\lambda_0$$

Thus

$$\Gamma_0^{(4)}(s,t,u) = -iZ_\lambda^{-1}\lambda_0 + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)$$

At the symmetric point we see that

$$\Gamma_0^{(4)}(s_0,t_0,u_0) = -iZ_\lambda^{-1}\lambda_0$$

with $\tilde{\Gamma}(s_0) = \tilde{\Gamma}(t_0) = \tilde{\Gamma}(u_0) = 0$. The renormalized 1PI 4 point function $\Gamma^{(4)}$ is related to Green's function by

$$\Gamma_R^{(4)} = \prod_{j=1}^{4}[i\Delta_R(p_j)]^{-1}G_R^{(4)}$$

which implies the following relation between renormalized and un-renormalized 1PI 4-point functions,

$$\Gamma_R^{(4)}(s,t,u) = Z_\phi^2\Gamma_0^{(4)}(s,t,u)$$

Or

$$\Gamma_R^{(4)}(s,t,u) = Z_\phi^2[-iZ_\lambda^{-1}\lambda_0 + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)]$$

Here we see that the divergences are contained in the first term, so we define the renormalized coupling constant λ by

$$\lambda = Z_\phi^2 Z_\lambda^{-1}\lambda_0 \tag{5}$$

In other words, we let the bare or unrenormalized coupling constant λ_0 be divergent in such a way that the combination $Z_\phi^2 Z_\lambda^{-1}\lambda_0$ is finite. The renormalized 1PI 4-point function is then

$$\Gamma_R^{(4)}(p_1,\cdots,p_4\ ) = Z_\phi^2\Gamma_0^{(4)} = -iZ_\lambda^{-1}Z_\varphi^2\lambda_0 + Z_\varphi^2[\tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)] = -i\lambda + Z_\varphi^2[\tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)]$$

Since $Z_\varphi = 1 + O(\lambda_0)$, $\tilde{\Gamma} = O(\lambda_0^2)$ $\lambda = \lambda_0 + O(\lambda_0^2)$, we can approximate

$$\Gamma_R^{(4)}(\ p_1,\cdots,p_4) = -i\lambda + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) + O(\lambda^3)$$

which is completely finite.

In Eq (5), we see that the unrenormalized coupling constant λ_0 has to diverge in such a way that the renomalized coupling constant λ is finite. Note that λ is measured in the physical processes involving $\Gamma_R^{(4)}$, i.e.

$$\Gamma_R^{(4)}(\ p_1,\cdots,p_4) = -i\lambda, \qquad \text{at} \quad \text{symmetric point} \quad s_0 = t_0 = u_0 = \frac{4\mu^2}{3}$$

♠Here we will illustrate the fact different normalization schemes will give the same physical result. Suppose we make the Taylor expansion about some other point, $s = s_1, t = t_1, u = u_1$, with $s_1 + t_1 + u_1 = 4\mu^2$. Then it is not hard to see that the renomalized 4 point function is of the form

$$\Gamma_R'^{(4)}(\ p_1,\cdots,p_4) = -i\lambda' + \tilde{\Gamma}'(s) + \tilde{\Gamma}'(t) + \tilde{\Gamma}'(u) + O(\lambda'^3)$$

where

$$\tilde{\Gamma}'(s) = \Gamma(s) - \Gamma(s_1), \quad \tilde{\Gamma}'(t) = \Gamma(t) - \Gamma(t_1), \quad \tilde{\Gamma}'(u) = \Gamma(u) - \Gamma(u_1)$$

and

$$-i\lambda_0 + \Gamma(s_1) + \Gamma(t_1) + \Gamma(u_1) = -iZ_\lambda'^{-1}\lambda_0, \qquad \lambda = Z_\phi^2 Z_\lambda'^{-1}\lambda_0$$

Note that $\Gamma_R^{(4)}$ and $\Gamma_R'^{(4)}$ are obtained from $\Gamma_0^{(4)}$ by substracting out some constants. So if we consider the angular distribution in the scattering reaction, these two functions $\Gamma_R^{(4)}, \Gamma_R'^{(4)}$ will have the same shape and can only differ by a constant. Suppose we write the angular distribution as

$$\Gamma_R^{(4)}(\, p_1, \cdots, p_4) = -i\lambda + f_1(\theta), \qquad \Gamma_R'^{(4)}(\, p_1, \cdots, p_4) = -i\lambda' + f_1(\theta) + b$$

where b is some contant. Since we need to compare the computed renomalized amplitudes $\Gamma_R^{(4)}, \Gamma_R'^{(4)}$ with some experimental measurement, say at $\theta = \theta_0$, in order to determine the coupling contants λ and λ', we get

$$-i\lambda + f_1(\theta_0) = -i\lambda' + f_1(\theta_0) + b, \qquad \text{or} \qquad -i\lambda = -i\lambda' + b$$

This shows that these two amplitudes $\Gamma_R^{(4)}, \Gamma_R'^{(4)}$ from 2 different renormalization schemes are the same.♠

After the mass, wave function and coupling constant renormalization, we can rewrite the original Lagrangian (unrenormalized Lagrangian)

$$\mathcal{L}_0 = \frac{1}{2}[(\partial_\mu \phi_0)^2 - \mu_0^2 \phi_0^2] - \frac{\lambda_0}{4!}\phi^4$$

in terms of renormalized quantities and counterterms

$$\mathcal{L}_0 = \mathcal{L} + \Delta\mathcal{L} \tag{6}$$

where

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \phi)^2 - \mu^2 \phi^2] - \frac{\lambda}{4!}\phi^4$$

is called the renormalized Lagrangian and

$$\Delta\mathcal{L} = \mathcal{L}_0 - \mathcal{L} = \frac{1}{2}(Z_\phi - 1)[(\partial_\mu \phi)^2 - \mu^2\phi^2] + \frac{\delta\mu^2}{2}\phi^2 - \frac{-\lambda(Z_\lambda - 1)}{4!}\phi^4$$

are called the counterterms. Here we have used the relations,

$$\mu^2 = \delta\mu^2 + \mu_0^2 \quad , \quad \phi = Z_\phi^{-\frac{1}{2}}\phi_0 \quad , \quad \lambda = Z_\lambda^{-1} Z_\phi^2 \lambda_0$$

1.2 BPH renormalization

An equivalent, perhaps more organized way of carrying out renormalization is the BPH (Bogoliubov, Parasiuk and Hepp) renormalization scheme. The essential idea here is to use the counter terms Lagrangian $\Delta\mathcal{L}$ as a device to cancel the divergences. In this scheme, the relation between lower divergences and high order one is more transparent. This scheme also provides a simple criterion for deciding whether a given theory is renormalizable or not. We will simply illustrate this procedure and discuss some simple points.

(a) Starts with renormalized Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

Generate free propagator and vertices from this Lagrangian.

(b) The divergent parts of one-loop 1PI diagrams are isolated by Taylor expansion. Construct a set of counter terms $\Delta\mathcal{L}^{(1)}$ to cancel these divergences.

(c) A new Lagrangian $\mathcal{L}^{(1)} = \mathcal{L} + \Delta\mathcal{L}^{(1)}$ is used to generate 2-loop diagrams and counter terms $\Delta\mathcal{L}^{(2)}$ to cancel 2-loops divergences. This sequence of operation is iteratively applied.
To illustrate the usefulness of BPH scheme, we need to make use of the following power counting method.

1.2.1 Power counting

Superficial degree of divergence D is defined as

$$D = (\sharp \ \ of\ loop\ momenta\ in\ numerator) - (\sharp \ \ of\ loop\ momenta\ in\ denominator)$$

We define the following quantities,
B= number of external lines
IB= number of internal lines
n= number of vertices
Counting *all* the lines in the graph, we get

$$4n = 2(IB) + B$$

When we compute the Feynman diagrams we need to integrate over the momentum of each internal line, subjected to the 4-momentum conseravation at each vertex But there is an overall 4-momentum conseravation which do not depend on the internal momentum. Then the number of loops, or unrestricted intenal momenta, L is given by

$$L = IB - n + 1$$

The superficial degree of divergence is

$$D = 4L - 2(IB)$$

Eliminating n, L and (IB), we get

$$D = 4 - B$$

Thus $D < 0$ for $B > 4$.Note that in this case D is independent of n, the order of pertubation, and depends only on B, the number of external lines. This means that all the superficial divergences reside in some small number of Green's functions. The $\lambda\phi^4$ theory has the symmetry $\phi \to -\phi$. which implies that $B = even$ and only $B = 2, 4$ are superficially divergent.

We now discuss these cases separately.

(a) $B = 2, \Rightarrow D = 2$, 2-point function–self energy graph

Being quadratically divergent, the necessary Taylor expansion for the 2-point function is of the form,

$$\Sigma\left(p^2\right) = \Sigma\left(0\right) + p^2\Sigma'\left(0\right) + \widetilde{\Sigma}\left(p^2\right)$$

where $\Sigma(0)$ and $\Sigma'(0)$ are divergent and $\widetilde{\Sigma}\left(p^2\right)$ is finite. Here for conveience we do the Taylor expansion around $p^2 = 0$. To cancel these divergences we need to add two counterterms to the Lagrangian,

$$\frac{1}{2}\Sigma\left(0\right)\phi^2 + \frac{1}{2}\Sigma'\left(0\right)\left(\partial_\mu\phi\right)^2$$

which give the following contributions to the self energy graphs,

Fig 5 Counter term for 2 point function

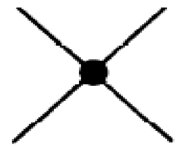
(b) $B = 4, \Rightarrow \ D = 0$, *4-point function–vertex graphs*

The Taylor expansion is

$$\Gamma^{(4)}\left(p_i\right) = \Gamma^{(4)}\left(0\right) + \widetilde{\Gamma}^{(4)}\left(p_i\right)$$

where $\Gamma^{(4)}(0)$ is logarithmically divergent which is to be cancelled by conunterterm of the form

$$\frac{i}{4!}\Gamma^{(4)}\left(0\right)\phi^4$$



$-\Gamma^{(4)}(0)$

Fig 6 counter term for 4-point function

To this order the general counterterrm Lagrangian is then

$$\Delta\mathcal{L} = \frac{1}{2}\Sigma(0)\,\phi^2 + \frac{1}{2}\Sigma'(0)\,(\partial_\mu\phi)^2 + \frac{i}{4!}\Gamma^{(4)}(0)\,\phi^4$$

Note that with the counter terms the Lagrangian is of the form

$$\begin{aligned}\mathcal{L}+\Delta\mathcal{L} &= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{1}{2}\Sigma(0)\,\phi^2 + \frac{1}{2}\Sigma'(0)\,(\partial_\mu\phi)^2 + \frac{i}{4!}\Gamma^{(4)}(0)\,\phi^4 \\ &= \frac{1}{2}\left(1+\Sigma'(0)\right)(\partial_\mu\phi)^2 - \frac{1}{2}\left(\mu^2+\Sigma(0)\right)\phi^2 - \frac{\lambda}{4!}\left(1+\Gamma^{(4)}(0)\right)\phi^4\end{aligned}$$

This clearly has the same structure as the unrenormalized Lagrangian. More specifically, if we define

$$\begin{aligned}&1+\Sigma'(0) = Z_\varphi, \qquad \phi_0 = Z_\varphi\phi \\ &\mu_0^2 = \left(\mu^2+\Sigma(0)\right)Z_\varphi^{-1} \\ &\lambda_0 = \lambda\left(1+\Gamma^{(4)}(0)\right)Z_\phi^{-2} = Z_\lambda Z_\phi^{-2}\lambda\end{aligned}$$

then we get

$$\mathcal{L}+\Delta\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_0)^2 - \frac{\mu_0^2}{2}\phi_0^2 - \frac{\lambda_0}{4!}\phi_0^4 \tag{7}$$

which is exactly the unrenormalized Lagrangain. Thus the BPH scheme is equivalent to the conventional substraction scheme but organized differently.

1.2.2 Analysis of divergences

The BPH renormalization scheme looks simple. It is remarkable that this simple scheme can serve as the basis for setting up a proof for a certain class of field theory. There are many interesting and useful features in BPH which do not show themselves on the first glance and are very useful in the understanding of this renormalization program. We will now discuss some of them.

(a) **Convergence of Feynman diagrams**

In our analysis so far, we have used the superficial degree of divergences D. It is clear that to 1-loop order that superficial degree of divergence is the same as the real degree of divergence. When we go beyond 1-loop it is possible to have an overall $D < 0$ while there are real divergences in the subgraphs. The real convergence of a Feynman graph is governed by *Weinberg's theorem* ([?]) : The general Feynman integral converges if the superficial degree of divergence of the graph together with the superficial degree of divergence of all subgraphs are negative. To be more explicit, consider a Feynman graph with n external lines and l loops. Introduce a cutoff Λ in the momentum integration to estimate the order of divergence,

$$\Gamma^{(n)}(p_1,\cdots,p_{n-1}) = \int_0^\Lambda d^4q_1\cdots d^4q_i I(p_1,\cdots,p_{n-1};q_1,\cdots,q_i)$$

Take a subset $S=\{q_1',q_2',\cdots q_m'\}$ of the loop momenta $\{q_1,\cdots,q_i\}$ and scale them to infinity and all other momenta fixed. Let $D(S)$ be the superficial degree of divergence associated with integration over this set, i. e.,

$$\left|\int_0^\Lambda d^4q_1'\cdots d^4q_m' I\right| \leq \Lambda^{D(s)}\{\ln\Lambda\}$$

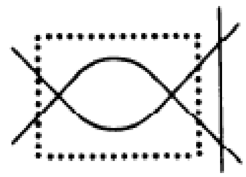


Figure 1: Fig 7 Divergent 6-point function

where $\{\ln \Lambda\}$ is some function of $\ln \Lambda$. Then the convergent theorem states that the integral over $\{q_1, \cdots, q_i\}$ converges if the $D(S)' s$ for all possible choice of S are negative. For example the graph in the following figureis a 6-point function with $D = -2$. But the integration inside the box with $D = 0$ is logarithmically divergent. However, in the BPH procedure this subdivergence is in fact removed by lower order counter terms as shown below.

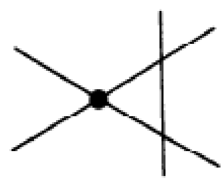


Fig 8 Counterterm for 6-point function

This illustrates that a graph with $D < 0$, may contain subgraphs which are divergent. But in BPH scheme, these divergences are removed by contributions coming from the lower order counterterms. This is why we can focus on those graphs with $D \geq 0$ and do not have to worry about divergences from the subgraphs.

(b) Primitively divergent graphs

A primitively divergent graph has a nonnegative overall superficial degree of divergence but is convergent for all subintegrations. Thus these are diagrams in which the only divergences is caused by all of the loop momenta growing large together. This means that when we differentiate with respect to external momenta at least one of the internal loop momenta will have more power in the denominator and will improve the convergence of the diagram. It is then clear that all the divergences can be isolated in the first few terms of the Taylor expansion. In othe words, if the diagrams are not primitively divergent, in the Taylor expansion, the differentiation with respect to external momenta may or may not improve the convergences of the graph.

(c) Disjointed divergent graphs

Here the divergent subgraphs are disjointed. For illustration, consider the 2-loop graph given below,

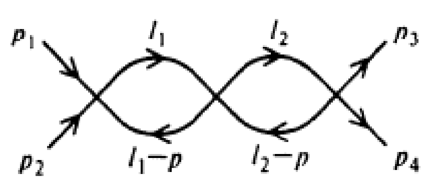


Fig 9 2-loop disjoint divergence

Here we have 2 separate quadratically divergent graphs and is certainly not a primatively divergent graph. It is clear that differentiating with respect to the external momentum will improve only one of the loop integration but not both. As a result, not all divergences in this diagram can be removed by subtracting out the first few terms in the Taylor expansion around external momenta. However, the lower order counter terms in the BPH scheme will come in to save the day. The Feynman integral is written as

$$\Gamma_a^{(4)}(p) \propto \lambda^3 [\Gamma(p)]^2$$

with

$$\Gamma(p) = \frac{1}{2}\int d^4 l \frac{1}{l^2 - \mu^2 + i\varepsilon} \frac{1}{\left[(l-p)^2 - \mu^2 + i\varepsilon\right]}$$

and $p = p_1 + p_2$. Since $\Gamma(p)$ is logarithmically divergent, $\Gamma_a^{(4)}(p)$ cannot be made convergent no matter how many derivatives act on it, even though the overall superficial degree of divergence is zero. However, we have the lower order counterm $-\lambda^2\Gamma(0)$ corresponding to the substraction introduced at the 1-loop level. This generates the additional contributions given in the following diagrams,

Fig 10 2-loop graphs with counterms

which are proportional to $-\lambda^3\Gamma(0)\Gamma(p)$. Adding these 3 contributions, we get

$$\begin{aligned} &\lambda^3 [\Gamma(p)]^2 - 2\lambda^3\Gamma(0)\Gamma(p) \\ &= \lambda^3 [\Gamma(p) - \Gamma(0)]^2 - \lambda^3 [\Gamma(0)]^2 \end{aligned}$$

Since the combination in the first $[\cdots]$ is finite, the divergence in the last term can be removed by one differentiation. Here we see that with the inclusion of lower order counterterms, the divergences take the form of polynomials in external momenta. Thus for graphs with disjointed divergences we need to include the lower order counter terms to remove the divergences by substractions in Taylor expansion.

(d) Nested divergent graphs

(a) In this case, one of a pair of divergent 1PI is entirely contained within the other as shown in the following diagram,

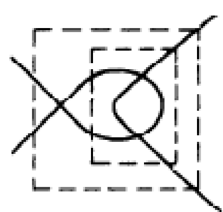


Fig 11 Nested divergence

After the subgraph divergence is removed by diagrams with lower order counterterms, the overall divergences is then renormalized by a λ^3 counter terms as shown below,

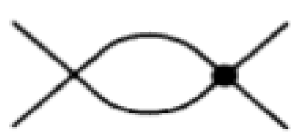


Fig 12 counterterm for nested divergence

Again diagrams with lower-order counterterm insertions must be included in order to aggregate the divergences into the form of polynomial in external momenta.

(e) Overlapping divergent graphs

These diagrams are those divergences which are neither nested nor disjointed. These are most difficult to analyze. An example of this is shown below,

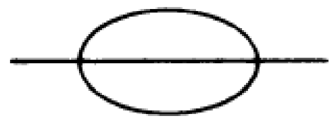


Fig 13 Overlapping divergence

The study of how to disentangle these overlapping divergences is beyond the scope of this simple introduction and we refer interested readers to the literature ([**?**].[**?**]).

From these discussion, it is clear that BPH renormalization scheme is quite useful in organizing the higher order divergences in a more systematic way for the removing of divergences by constructing the counterterms. In particular, the lower order counter terms play an important role in removing divergences in the subgraphs.

The general analysis of the renormalization program has been carried out by Bogoliubov, Parasiuk, Hepp ([**?**]). The result is known as BPH theorem, which states that for a general renormalizable field theory, to any order in perturbation theory, all divergences are removed by the counterterms corresponding to superficially divergent amplitudes.

1.3 Regularization

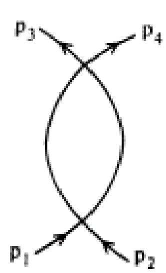
In carrying out the renormalization, we need first to make divergent integral finite before we can do any *mathematically meaningful* manipulation. There are two different schemes frequently used to make the integrals finite, Pauli-Villars regularization and dimensional regularization. The latter one is a very powerful method for dealing with theories with symmetries and is used widely in the calculation in gauge theories.

1.3.1 Pauli-Villars Regularization

In this scheme, we repalce the propagator by the one where we substract from it another propagator with very large mass,

$$\frac{1}{k^2-\mu_0^2} \rightarrow (\frac{1}{k^2-\mu_0^2}-\frac{1}{k^2-\Lambda^2})=\frac{(\mu_0^2-\Lambda^2)}{(k^2-\mu_0^2)(k^2-\Lambda^2)} \rightarrow \frac{1}{k^4} \quad \text{for large } k$$

which will make the integral more convergent. This has the advantage of being covariant as compared to simply cutting off the integral at some large momenta. We will illustrate this by an example of 4-point function from the following graph,



whic gives,

$$\Gamma\left(p^2\right)=\Gamma\left(s\right)=\frac{\left(-i\lambda\right)^2}{2}\int\frac{d^4l}{\left(2\pi\right)^4}\frac{i}{\left(l-p\right)^2-\mu^2}\frac{i}{l^2-\mu^2}$$

With Pauli-Villars regularization this becomes,

$$\Gamma\left(p^2\right) = \frac{-\lambda^2\Lambda^2}{2}\int\frac{d^4l}{(2\pi)^4}\frac{1}{\left[(l-p)^2-\mu^2\right](l^2-\mu^2)(l^2-\Lambda^2)}$$

Taylor expansion around $p^2=0$, gives

$$\Gamma\left(p^2\right) = \Gamma(0) + \widetilde{\Gamma}\left(p^2\right)$$

with

$$\Gamma(0) = \frac{-\lambda^2\Lambda^2}{2}\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-\mu^2)^2(l^2-\Lambda^2)}$$

and

$$\widetilde{\Gamma}\left(p^2\right) = \frac{-\lambda^2\Lambda^2}{2}\int\frac{d^4l}{(2\pi)^4}\frac{2l\cdot p-p^2}{\left[(l-p)^2-\mu^2\right](l^2-\mu^2)^2(l^2-\Lambda^2)}$$

Since the integral in $\widetilde{\Gamma}\left(p^2\right)$ is convergent enough, we can take the limit $\Lambda^2\to\infty$ inside the integral to get,

$$\widetilde{\Gamma}\left(p^2\right) = \frac{\lambda^2}{2}\int\frac{d^4l}{(2\pi)^4}\frac{2l\cdot p-p^2}{\left[(l-p)^2-\mu^2\right](l^2-\mu^2)^2}$$

Note that in $\widetilde{\Gamma}\left(p^2\right)$ we have take the limit $\Lambda^2\to\infty$ inside the integral because it is a sufficiently convergent integral. The standard method in the computation of the Feynman graphs is to combine the denominators by using the identities,

$$\frac{1}{a_1a_2\cdots a_n} = (n-1)!\int_0^1\frac{dz_1dz_2\cdots dz_n}{(a_1z_1+\cdots+a_nz_n)^n}\delta\left(1-\sum_{i=1}^n z_i\right)$$

$$\frac{1}{a_1^2a_2\cdots a_n} = n!\int_0^1\frac{z_1dz_1dz_2\cdots dz_n}{(a_1z_1+\cdots+a_nz_n)^{n+1}}\delta\left(1-\sum_{i=1}^n z_i\right)$$

Here $\alpha_1,\cdots\alpha_n$ are usually called the Feynman parameters. ♠We will illustrate the derivation of this identity as follows. We first verify the simple case of $n=2$,

$$\begin{aligned}\int_0^1\frac{dz_1dz_2\delta(1-z_1-z_2)}{(a_1z_1+a_2z_2)^2} &= \int_0^1\frac{dz_1}{[a_1z_1+a_2(1-z_1)]^2} = \int_0^1\frac{dz_1}{[(a_1-a_2)z_1+a_2]^2}\\ &= \frac{1}{(a_1-a_2)}\left[\frac{-1}{(a_1-a_2)z_1+a_2}\right]_0^1 = \frac{1}{a_1a_2}\end{aligned}$$

Differentiate this relation with respect to a_1,

$$\frac{1}{a_1^2a_2} = 2\int_0^1\frac{z_1dz_1dz_2\delta(1-z_1-z_2)}{(a_1z_1+a_2z_2)^3}$$

Then

$$\frac{1}{a_1a_2a_3} = \frac{1}{a_3}\int_0^1\frac{dz_1dz_2\delta(1-z_1-z_2)}{(a_1z_1+a_2z_2)^2}$$

Apply this to the combination $\dfrac{1}{(a_1z_1+a_2z_2)^2}$ and $\dfrac{1}{a_3}$,

$$\begin{aligned}I &= \int\frac{dz_1dz_2\delta(1-z_1-z_2)}{(a_1z_1+a_2z_2)^2}\frac{1}{a_3} = 2\int\frac{y_2dy_1dy_2\delta(1-y_1-y_2)}{[y_1a_3+y_2(a_1z_1+a_2z_2)]^3}dz_1dz_2\delta(1-z_1-z_2)\\ &= 2\int\frac{(1-y_1)dy_1dz_1dz_2\delta(1-z_1-z_2)}{[y_1a_3+(1-y_1)(a_1z_1+a_2z_2)]^3}\end{aligned}$$

Now define

$$(1-y_1)\,z_1 = x_1, \qquad (1-y_1)\,z_2 = x_2$$

we get

$$\begin{aligned} I &= 2\int \frac{(1-y_1)\,dy_1 dz_1 dz_2 \delta(\dfrac{1-y_1-x_1-x_2}{1-y_1})}{[y_1 a_3 + (a_1 x_1 + a_2 x_2)]^3}\left(\frac{1}{1-y_1}\right)^2 \\ &= 2\int \frac{dy_1 dz_1 dz_2 \delta\,(1-y_1-x_1-x_2)}{[y_1 a_3 + (a_1 x_1 + a_2 x_2)]^3} \end{aligned}$$

Or

$$\frac{1}{a_1 a_2 a_3} = 2\int \frac{dz_3 dz_1 dz_2 \delta\,(1-x_3-x_1-x_2)}{[z_3 a_3 + (a_1 x_1 + a_2 x_2)]^3}$$

where we have renamed y_1 as x_3. This verifies the case for combining 3 factors $a_1 a_2 a_3$ in the denominator. From this we can see how to verify the most general cases.

Using these relations we get,♠

$$\frac{1}{\left[(l-p)^2-\mu^2\right](l^2-\mu^2)^2} = 2\int \frac{(1-\alpha)\,d\alpha}{A^3}$$

where

$$A = (1-\alpha)\left(l^2-\mu^2\right) + \alpha\left[(l-p)^2-\mu^2\right] = (l-\alpha p)^2 - a^2$$

with

$$a^2 = \mu^2 - \alpha\,(1-\alpha)\,p^2$$

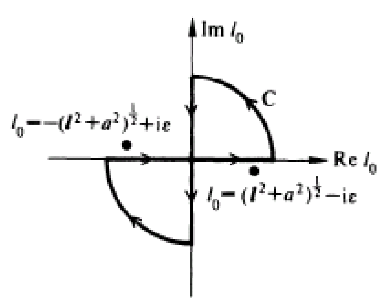
Thus

$$\begin{aligned} \widetilde{\Gamma}\left(p^2\right) &= \lambda^2 \int_0^1 (1-\alpha)\,d\alpha \int \frac{d^4 l}{(2\pi)^4} \frac{2l\cdot p - p^2}{\left[(l-\alpha p)^2 - a^2\right]^3} \\ &= \lambda^2 \int_0^1 (1-\alpha)\,d\alpha \int \frac{d^4 l}{(2\pi)^4} \frac{(2\alpha-1)\,p^2}{(l^2-a^2+i\varepsilon)^3} \end{aligned}$$

where we have changed the variable $l \to l + \alpha p$ and drop the term linear in l. This integral now depends only on l^2 and not on any angles. But it is more conveient to transform this integral into integration in Euclean space rather than the Minkowski spce. To carry out this, we note that in the complex l_0 plane, the integrand has poles at

$$l_0 = \pm\left[\sqrt{l^2+a^2} - i\varepsilon\right]$$

As shown in the graphs we can rotate the contour of the integration by 90^0 by the following procedure (*Wick rotation*).



From Cauchy's theorem we have

$$\oint_C dl_0 f(l_0) = 0$$

where

$$f(l_0) = \frac{1}{\left[l_0^2 - (\sqrt{\vec{l}^{\,2} + a^2} - i\varepsilon)^2\right]^3}$$

Since $f(l_0) \to l_0^{-6}$ as $l_0 \to \infty$, the contribution from the circular part of contour C with very large radius vanishes and we get

$$\int_{-\infty}^{\infty} dl_0 f(l_0) = \int_{-i\infty}^{i\infty} dl_0 f(l_0)$$

Thus the integration path has been rotated from along real axis to imaginary axis (Wick rotation). Changing the variable $l_0 = il_4$, so that l_4 is real we found

$$\int_{-i\infty}^{i\infty} dl_0 f(l_0) = i\int_{-\infty}^{\infty} dl_4 f(il_4) = -i\int_{-\infty}^{\infty} \frac{dl_4}{\left(l_1^2 + l_2^2 + l_3^2 + l_4^2 + a^2 - i\varepsilon\right)^3}$$

Define the Euclidean momentum as $k_i = (l_1, l_2, l_3, l_4)$ with $k^2 = l_1^2 + l_2^2 + l_3^2 + l_4^2$. The integral is then

$$\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - a^2 + i\varepsilon)^3} = -i\int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 + a^2 - i\varepsilon)^3}$$

Using *spherical* coordinates in 4-dimensional Euclidean space, we have

$$\int d^4 k = \int_0^{\infty} k^3 dk \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{\pi} \sin^2\chi d\chi$$

and integrating over angles we found

$$\begin{aligned} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 + a^2 - i\varepsilon)^3} &= 2\pi^2 \int_0^{\infty} \frac{k^3 dk}{(2\pi)^4} \frac{1}{(k^2 + a^2 - i\varepsilon)^3} \\ &= \frac{1}{16\pi^2} \int_0^{\infty} \frac{k^2 dk^2}{(k^2 + a^2 - i\varepsilon)^3} \end{aligned}$$

Many intergrals in loop integration can be worked out using Gamma and Beta functions. The standard Gamma function is defined by

$$\Gamma(n) = \int_0^{\infty} u^{n-1} e^{-u} du \tag{8}$$

It follows that

$$\Gamma(m)\Gamma(n) = \int_0^{\infty} u^{n-1} e^{-u} du \int_0^{\infty} v^{m-1} e^{-v} dv$$

Now we transform this into 2-dimensional integration by setting $u = x^2, v = y^2$,

$$\Gamma(m)\Gamma(n) = 4\int dx dy e^{-\left(x^2 + y^2\right)} x^{2n-1} y^{2m-1}$$

Using polar coordinates, $x = r\cos\theta, y = \sin\theta$, we get

$$\begin{aligned} \Gamma(m)\Gamma(n) &= 4\int dr e^{-r^2} r^{2(n+m)-1} \int_0^{\pi/2} d\theta (\cos\theta)^{2n-1} (\sin\theta)^{2m-1} \\ &= 2\Gamma(m+n) \int_0^{\pi/2} d\theta (\cos\theta)^{2n-1} (\sin\theta)^{2m-1} \end{aligned}$$

Thus we get the general integrals over trigonometric functions,

$$\int_0^{\pi/2} d\theta\,(\cos\theta)^{2n-1}(\sin\theta)^{2m-1} = \frac{1}{2}\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \frac{1}{2}B(n,m)$$

Or

$$\int_0^{\pi/2} d\theta\,(\cos\theta)^{n}(\sin\theta)^{m} = \frac{1}{2}B\left(\frac{n+1}{2},\frac{m+1}{2}\right) \tag{9}$$

where $B(n,m)$ is the usual Beta function. Let $u=\cos^2\theta$, we get

$$B(n,m) = \int_0^1 u^{m-1}(1-u)^{n-1}\,du$$

$u=x^2$,

$$B(n,m) = 2\int_0^1 x^{2m-1}\left(1-x^2\right)^{n-1}du$$

Let $t=\dfrac{x^2}{1-x^2}$

$$B(n,m) = \int_0^\infty \frac{t^{m-1}dt}{(1+t)^{m+n}} = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \tag{10}$$

This is the formula used in the computation of the Feynman integral.

Using this formula in Eq(10), we get

$$\int \frac{t^{m-1}dt}{(t+a^2)^n} = \frac{1}{(a^2)^{n-m}}\frac{\Gamma(m)\,\Gamma(n-m)}{\Gamma(n)}$$

For our integral the result is

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+a^2-i\varepsilon)^3} = \frac{1}{32\pi^2(a^2-i\varepsilon)}$$

and

$$\widetilde{\Gamma}(p^2) = \frac{-i\lambda^2}{32\pi^2}\int_0^1 \frac{d\alpha\,(1-\alpha)(2\alpha-1)\,p^2}{[\mu^2-\alpha(1-\alpha)\,p^2-i\varepsilon]}$$

It is straightforward to carry out the integration over the Feynman parameter α to get

$$\begin{aligned}
\tilde{\Gamma}(p^2) &= \tilde{\Gamma}(s) = \frac{i\lambda^2}{32\pi^2}\left\{2+\left(\frac{4\mu^2-s}{|s|}\right)^{\frac{1}{2}}\ln\left[\frac{(4\mu^2-s)^{\frac{1}{2}}-(|s|)^{\frac{1}{2}}\}}{\{(4\mu^2-s)^{\frac{1}{2}}+(|s|)^{\frac{1}{2}}\}}\right]\right\} && \text{for } s<0\\
&= \frac{i\lambda^2}{32\pi^2}\left\{2-2\left(\frac{4\mu^2-s}{s}\right)^{\frac{1}{2}}\tan^{-1}\left(\frac{s}{4\mu^2-s}\right)^{\frac{1}{2}}\right\} && \text{for } 0<s<4\mu^2\\
&= \frac{i\lambda^2}{32\pi^2}\left\{2+(\frac{s-4\mu^2}{s})^{\frac{1}{2}}ln\left[\frac{s^{\frac{1}{2}}-(s-4\mu^2)^{\frac{1}{2}}}{s^{\frac{1}{2}}+(s-4\mu^2)^{\frac{1}{2}}}\right]+i\pi\right\} && \text{for } s>4\mu^2
\end{aligned}$$

Using the same procedure, we can calculate the divergent term to give

$$\Gamma(0) = \frac{i\lambda^2\Lambda^2}{32\pi^2}\int d\alpha\frac{\alpha}{\alpha(\mu^2-\Lambda^2)+\Lambda^2} \simeq \frac{i\lambda^2}{32\pi^2}\ln\frac{\Lambda^2}{\mu^2}, \qquad \text{for large } \Lambda$$

1.3.2 Dimensional regularization

The basic idea here is that since the divergences come from integration of internal momentum in 4-dimensional space, the integral can be made finite in lower dimensional space. We can find a way to define the Feynman integrals as functions of space-time n and carry out the renormalization for lower values of n before analytically continuing in n *and* taking the limit $n \to 4$. We will illustrate this by an example.

Consider the integral

$$I = \int \frac{d^4k}{(2\pi)^4}(\frac{1}{k^2-\mu^2})[\frac{1}{(k-p)^2-\mu^2}]$$

which is logarithmically divergent in 4-dimension. If we define this as integration over $n-$dimension

$$I(n) = \int \frac{d^nk}{(2\pi)^4}\frac{1}{(k^2-\mu^2)}\left[\frac{1}{(k-p)^2-\mu^2}\right]$$

then the integral is convergent for $n<4$.To define this integral for non-integer values of n, we first combine the denominators using Feynman parameters and make the Wick rotation,

$$\begin{aligned} I(n) &= \int_0^1 d\alpha \int \frac{d^nk}{\left[(k-\alpha p)^2 - a^2 + i\varepsilon\right]^2} \\ &= i\int_0^1 d\alpha \int \frac{d^nk}{[k^2+a^2-i\varepsilon]^2} \qquad \text{with} \quad a^2 = \mu^2 - \alpha(1-\alpha)p^2 \end{aligned}$$

Now introduce the spherical coordinates in n-dimension and carry out the angular intgrations,

$$\begin{aligned} \int d^nk &= \int_0^\infty k^{n-1}dk \int_0^{2\pi} d\theta_1 \int_0^\pi \sin\theta_2 d\theta_2 \int \cdots \int_0^\pi \sin^{n-2}\theta_{n-1}d\theta_{n-1} \\ &= \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\int_0^\infty k^{n-1}dk \end{aligned}$$

where we have used the formula derived before,

$$\int_0^\pi \sin^m\theta d\theta = \frac{\sqrt{\pi}\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}$$

where we have used

$$\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

The the $n-$dimensional integral is

$$I(n) = \frac{2i\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}\int_0^1 d\alpha \int_0^\infty \frac{k^{n-1}dk}{[k^2+a^2-i\varepsilon]^2}$$

The dependence on n is now explicit and the integral is well-defined for $0 < \mathrm{Re}(n) < 4$. We can extend this domain of analyticity by integration by parts

$$\frac{1}{\Gamma(\frac{n}{2})}\int_0^\infty \frac{k^{n-1}dk}{[k^2+a^2-i\varepsilon]^2} = \frac{-2}{\Gamma(\frac{n}{2}+1)}\int_0^\infty k^n dk \frac{d}{dk}\left(\frac{1}{[k^2+a^2-i\varepsilon]^2}\right)$$

where we have used

$$z\Gamma(z) = \Gamma(z+1)$$

The integral is now well defined for -2 < Re(n) < 4. Repeat this procedure m times, the analyticity domain is extended to $-2m <$ Re$(n) < 4$ and eventually to Re$(n) \to -\infty$. To see what happens as $n \to 4$, we can integrate over k to get

$$I(n) = i\pi^{n/2}\Gamma\left(2 - \frac{n}{2}\right)\int_0^1 \frac{d\alpha}{[a^2 - i\varepsilon]^{2-n/2}}$$

Using the formula,

$$\Gamma\left(2 - \frac{n}{2}\right) = \frac{\Gamma\left(3 - \frac{n}{2}\right)}{2 - \frac{n}{2}} \to \frac{2}{4-n} \quad as\ n \to 4$$

we see that the singularity at $n = 4$ is a simple pole. Expand everything around $n = 4$,

$$\Gamma\left(2 - \frac{n}{2}\right) = \frac{2}{4-n} + \gamma_E + (n-4)B + \cdots$$

$$a^{n-4} = 1 + (n-4)\ln a + \cdots$$

where γ_E is the Euler constant and B is some constant, we obtain the limit, as $n \longrightarrow 4$

$$I(n) \longrightarrow \frac{2i\pi^2}{4-n} - i\pi^2\int_0^1 d\alpha \ln[\mu^2 - \alpha(1-\alpha)p^2] + i\pi^2\gamma$$

and the 1-loop contribution to 4-point function is,

$$\Gamma(p^2) = \frac{\lambda^2}{32\pi^2}\left\{\frac{2i}{4-n} - i\int_0^1 d\alpha \ln[\mu^2 - \alpha(1-\alpha)p^2] + iA\right\}$$

Taylor expansion around $p^2 = 0$ gives

$$\Gamma(p^2) = \Gamma(0) - \tilde{\Gamma}(p^2)$$

$$\Gamma(0) = \frac{\lambda^2}{32\pi^2}\left(\frac{2i}{4-n} - i\ln\mu^2 + iA\right) \simeq \frac{i\lambda^2}{16\pi^2(4-n)}$$

and

$$\begin{aligned}\tilde{\Gamma}(p^2) &= \frac{-i\lambda^2}{32\pi^2}\int_0^1 d\alpha \ln\left[\frac{\mu^2 - \alpha(1-\alpha)p^2}{\mu^2}\right] \\ &= \frac{-i\lambda^2}{32\pi^2}\int_0^1 \frac{d\alpha(1-\alpha)(2\alpha-1)p^2}{[\mu^2 - \alpha(1-\alpha)p^2]}\end{aligned}$$

Clearly this finite part is exactly the same as that given by the method of covariant regulariztion *(Pauli-Villars)*.

This method which we substract out first few terms of the Taylor expansion is usually referred to as the momentum subtraction scheme. But in the dimensional regularization seheme all the divergences show up as poles in $n = 4$. Then we can remove the divergences simply by substracting out the poles at $n = 4$. This is usually called the minimal subtraction scheme (MS). Usually,these pole at $n = 4$ are accompanies by terms involoving Euler constant and log (4π). One can substract these terms as well. This is known as the modified minimal subtraction or $\overline{MS}$.

The 1-loop self energy, which is quadraticlly divergent becomes in dimensional-regularization scheme,

$$-i\Sigma(p^2) = \frac{\lambda}{2}\int \frac{d^nk}{(2\pi)^4}\frac{1}{k^2 - \mu^2 + i\varepsilon} = \frac{-i\lambda\pi^{n/2}\Gamma\left(1 - \frac{n}{2}\right)}{32\pi^4(\mu^2)^{1-n/2}}$$

From the relation,

$$\Gamma\left(1 - \frac{n}{2}\right) = \frac{\Gamma\left(3 - \frac{n}{2}\right)}{\left(1 - \frac{n}{2}\right)\left(2 - \frac{n}{2}\right)}$$

we see that the quadratic divergnece has pole at $n = 4$ and also at $n = 2$. For $n \to 4$ we have,

$$-i\Sigma(0) = \frac{i\lambda\mu^2}{16\pi^2}\left(\frac{1}{4-n}\right)$$

2 Power counting and Renormalizability

We now discuss the problem of renormalization for more general interactions. It is clear that it is advantageous to use the BPH scheme in this discussion.

2.1 Theories with fermions and scalar fields

We first study the simple case of fermion ψ and scalar field ϕ. First consider some simple examples.

(a) 4-fermion interaction where the interaction Lagrangian is given by

$$\mathcal{L}_I = g\left(\bar{\psi}\psi\right)^2$$

Let

$$\begin{aligned} F &= \text{number of external fermion lines} \\ IF &= \text{number of internal fermion lines} \end{aligned}$$

Then from counting total numbers of fermion lines we have

$$F + 2\,(IF) = 4n$$

and *from the number of uncontrained momenta*

$$L = (IF) - n + 1$$

where n is the number of vertices. The superficial degree of divergence is then

$$D = 4L - (IF)$$

L and IF can eliminated to give a simple formula for D,

$$D = 4 - \frac{3}{2}F + 2n$$

This means that for n large enough $D > 0$ for any value of F and we need infinite *number of* different type of counterterms. The counter terms needed are

$$F = 2: \qquad \bar{\psi}\psi, \quad \bar{\psi}\partial\psi, \quad \bar{\psi}\partial\partial\psi, \quad \bar{\psi}\partial\partial\partial\psi, \quad \cdots$$

$$F = 4: \qquad \bar{\psi}\psi\bar{\psi}\psi, \quad \bar{\psi}\partial\psi\bar{\psi}\psi, \quad \bar{\psi}\partial\psi\bar{\psi}\partial\psi, \quad \bar{\psi}\psi\bar{\psi}\partial\partial\psi, \quad \bar{\psi}\partial\partial\partial\psi\psi\psi, \cdots$$

$$\cdots$$

Thus we will not be able to aborb these infinities by redefining the parameters in the Lagrangian.

(b) Yukawa interaction with interaction of the form,

$$\mathcal{L}_I = f\left(\bar{\psi}\gamma_5\psi\right)\phi$$

Then we have

$$F + 2\,(IF) = 2n, \qquad B + 2\,(IB) = n,$$

and

$$L = (IF) + (IB) - n + 1$$

where n is the number of vertices. The superficial degree of divergence is

$$D = 4L - (IF) - 2(IB)$$

and can *be* simplied to give

$$D = 4 - B - \frac{3}{2}F$$

Now D is independent of n. Thus $D \geq 0$, only for small number of cases and the counterterms needed are

$$B = 2, \qquad \phi^2, \qquad (\partial_\mu \phi)^2, \qquad B = 4, \qquad \phi^4$$

$$F = 2; \qquad \bar{\psi}\psi, \quad \bar{\psi}\partial\psi, \qquad F = 2,\ B = 1; \qquad \phi\bar{\psi}\psi,$$

So the counter terms can be absorbed in the redefinition of the parameters in the Lagrangian.

We now can study the most general cases. Write the Lagrangian density in the general form

$$\mathcal{L} = \mathcal{L}_0 + \sum_i \mathcal{L}_i$$

where $\mathcal{L}_0$ is the free Lagrangian quadratic in the fields and $\mathcal{L}_i$ are the interaction terms e.g.

$$\mathcal{L}_i = g_1 \bar{\psi}\gamma^\mu \psi \partial_\mu \phi, \qquad g_2 \left(\bar{\psi}\psi\right)^2, \qquad g_3 \left(\bar{\psi}\psi\right)\phi, \qquad \cdots$$

Here ψ denotes a fermion field and ϕ a scalar field. Define the following quantities

$$\begin{aligned}
n_i &= \text{number of } i-th \text{ type vertices} \\
b_i &= \text{number of scalar lines in } i-th \text{ type vertex} \\
f_i &= \text{number of fermion lines in } i-th \text{ type vertex} \\
d_i &= \text{number of derivatives in } i-th \text{ type of vertex} \\
B &= \text{number of external scalar lines} \\
F &= \text{number of external fermion lines} \\
IB &= \text{number of internal scalar lines} \\
IF &= \text{number of internal fermion lines}
\end{aligned}$$

Counting the *total* scalar and fermion lines, we get

$$B + 2\,(IB) = \sum_i n_i b_i \tag{11}$$

$$F + 2\,(IF) = \sum_i n_i f_i \tag{12}$$

Using momentum conservation at each vertex we can compute the number of loop integration L as

$$L = (IB) + (IF) - n + 1, \qquad n = \sum_i n_i$$

where the last term is due to the overall momentum conservation which does not contain the loop integrations. The superficial degree of divergence is then given by

$$D = 4L - 2\,(IB) - (IF) + \sum_i n_i d_i$$

Using the relations given in Eqs(11,12) we can write,

$$D = 4 - B - \frac{3}{2}F + \sum_i n_i \delta_i \tag{13}$$

where

$$\delta_i = b_i + \frac{3}{2}f_i + d_i - 4$$

is called the *index of divergence* of the interaction. Using the fact that Lagrangian density $\mathcal{L}$ has dimension 4 and scalar field, fermion field and the derivative have dimensions, 1, $\frac{3}{2}$, and 1 respectively, we get for the dimension of the coupling constant g_i as

$$\dim(g_i) = 4 - b_i - \frac{3}{2} f_i - d_i = -\delta_i$$

Thus the index of divergence of each interaction term is related to the dimension of the corresponding coupling constant.

We distinguish 3 different situations;

(a) $\delta_i < 0$
In this case, D decreases with the number of i-th type of vertices and the interaction is called *super − renormalizable interaction*. The divergences occur only in some lower order diagrams. There is only one type of theory in this category, namely ϕ^3 interaction.

(b) $\delta_i = 0$
Here D is independent of the number of i-th type of vertices and interactions are called *renormalizable interactions. The divergence are present only in small number Green's functions.* Interactions in this category are of the form, $g\phi^4$, $f\left(\bar{\psi}\psi\right)\phi$.

(c) $\delta_i > 0$
Then D increases with the number of the i-th type of vertices and all Green's functions are divergent for large enough n_i. These are called *non−renormalizable interactions*. There are plenty of examples in this category, $g_1\bar{\psi}\gamma^\mu\psi\partial_\mu\phi$, $g_2\left(\bar{\psi}\psi\right)^2$, $g_3\phi^5, \cdots$

The index of divergence δ_i can be related to the operator's *canonical dimension* which is defined in terms of the high energy behavior of *the propagator* in the free field theory. More specifically, for any operator A, we write the 2-point function as

$$D_A\left(p^2\right) = \int d^4x e^{-ip\cdot x} \left\langle 0\left|T\left(A(x)A(0)\right)\right|0\right\rangle$$

If the asymptotic behavior is of the form,

$$D_A\left(p^2\right) \longrightarrow \left(p^2\right)^{-\omega_A/2}, \qquad as \qquad p^2 \longrightarrow \infty$$

then the canonical dimension is defined as

$$d(A) = (4 - \omega_A)/2$$

Thus for the case of fermion and scalar fields we have,

$$d(\phi) = 1, \qquad d(\partial^n\phi) = 1 + n$$

$$d(\psi) = \frac{3}{2}, \qquad d(\partial^n\psi) = \frac{3}{2} + n$$

Note that in these simple cases, these values are the same as those obtained in the dimensional analysis in the classical theory and sometimes they are also called the naive dimensions. As we will see later for the vector field, the canonical dimension is not necessarily the same as the naive dimension.

For composite operators that are polynomials in the scalar or fermion fields it is difficult to know their asymptotic behavior. So we define their canonical dimensions as the algebraic sum of their constituent fields. For example,

$$d\left(\phi^2\right) = 2, \qquad d\left(\bar{\psi}\psi\right) = 3$$

For general composite operators that show up in the those interaction described before, we have,

$$d(\mathcal{L}_i) = b_i + \frac{3}{2} f_i + d_i$$

and it is related to the index of divergence as

$$\delta_i = d(\mathcal{L}_i) - 4$$

We see that a dimension 4 interaction is renormalizable and greater than 4 is non-renormalizable. So if we require the theory to be renormalizable, then this will restrict the possible interaction greatly. Basically, for theories with scalar and fermion fields, they are of the type ϕ^3, ϕ^4, $\left(\bar{\psi}\psi\right)\phi$ and their generalization to include internal symmetry indicies.

Counter terms

Recall that we add counterterms to cancel all the divergences in Green's functions with superficial degree of divergences $D \geq 0$. For convenience we use the Taylor expansion around zero external momenta $p_i = 0$. It is easy to see that for a general $D \geq 0$ diagram with given F and B ,counter terms will be of the form

$$O_{ct} = (\partial_\mu)^\alpha (\psi)^F (\phi)^B, \qquad \alpha = 0, 1, 2, \cdots D \tag{14}$$

For example, in the Yukawa theory, the graph with $F = 2$, 2 fermio lines,has $D = 1$. So the counter terms required wil have $\alpha = 0, 1$, and are of the forms,

$$\bar{\psi}\psi, \qquad \bar{\psi}\gamma_\mu \partial^\mu \psi$$

The the canonical dimension of the counter term in Eq (14) is given by

$$d_{ct} = \frac{3}{2} F + B + \alpha$$

The index of divergence of the counterterms is

$$\delta_{ct} = d_{ct} - 4$$

Using the relation in Eq (13) we can write this as

$$\delta_{ct} = (\alpha - D) + \sum_i n_i \delta_i$$

Since $\alpha \leq D$, we have the result

$$\delta_{ct} \leq \sum_i n_i \delta_i$$

This means that the counterterms induced by a Feynman diagrams have indices of divergences less or equal to the sum of the indices of divergences of all interactions δ_i in the diagram. We then get the very useful result that the renormalizable interactions which have $\delta_i = 0$ will generate counterterms with $\delta_{ct} \leq 0$. Thus if all the $\delta_i \leq 0$ terms are present in the original Lagrangain, then the counter terms will have the same structure as the interactions in the original Lagragian and they may be considered as redefining parameters like masses and coupling constants in the theory. On the other hand non-renormalizable interactions which have $\delta_i > 0$ will generate counterterms with arbitrary large δ_{ct} in sufficiently high orders and clearly cannot be absorbed into the original Lagrangian by a redefinition of parameters δ_{ct}. Thus non-renormalizable theories will not necessarily be infinite; however the infinite number of counterterms associated with a non-renormalizable interaction will make it lack in predictive power and hence be unattractive, in the framework of perturbation theory.

We will adopt a more restricted definition of renormalizability: a Lagrangian is said to be renormalizable by power counting if all the counterterms induced by the renormalization procedure can be absorbed by redefinitions of parameters in the Lagrangian. With this definition the theory with Yukawa interaction $\bar{\psi}\gamma_5\psi\phi$ by itself, is not renormalizable even though the coupling constant is dimensionless. This is because the 1-loop diagram shown below

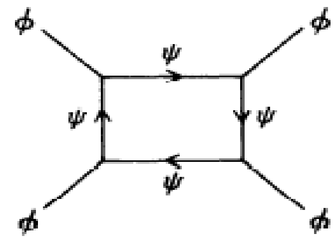


Fig 14 Box diagram for Yukawa

is logarithmically divergent and needs a counter term of the form ϕ^4 which is not present in the original Lagrangain. Thus Yukawa interaction with additional ϕ^4 interaction is renormalizable.

2.2 Theories with vector fields

Here we distinguish massless from massive vector fields because their asymptotic behaviors for the free field propagators are very different.

(a) Massless vector field
Massless vector field A_μ is usually associated with local gauge invariance as in the case of QED. The asymptotic behavior of free field propagator for such vector field is very similar to that of scalar field. For example, in the Feynman gauge we have

$$\Delta_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon} \longrightarrow O\left(k^{-2}\right), \qquad \text{for large } k^2$$

which has the same asymptotic behaior as that of scalar field. The canonical dimension is then

$$d\left(A_\mu\right) = 1$$

and is the same as the scalar field. Thus the power counting for theories with massless vector field interacting with fermions and scalar fields is the same as before. The renomalizable interactions in this category are of the type,

$$\bar{\psi}\gamma_\mu\psi A^\mu, \qquad \phi^2 A_\mu A^\mu, \qquad \left(\partial_\mu\phi\right)\phi A^\mu$$

Here A^μ is a massless vector field and ψ a fermion field.
♠Note that in this theory of massless vector field one needs to consider the gauge invariance in the construction of the counter terms. Consider the case of QED where the Lagragian is given by

$$\mathcal{L} = \overline{\psi}(x)\gamma^\mu\left(i\partial_\mu - eA_\mu\right)\psi(x) - m\overline{\psi}(x)\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

and is invariant under the gauge transformation,

$$A^\mu \to A'^\mu = A^\mu - \partial^\mu\alpha, \qquad \psi(x) \to \psi'(x) = e^{ie\alpha}\psi$$

In this theory the superficial degree of divergence is given by

$$D = 4 - B - \frac{3}{2}F$$

Thus for $D \geq 0$, we have the following cases,

(a) $B = 2$, $F = 0$, and $D = 2$
This means that $1PI$ 2-point Green's function for the photon field,

$$\pi_{\mu\nu}(q) = \int d^4x e^{-iq\cdot x}\left\langle 0\left|T\left(J_\mu(x)J_\nu(0)\right)\right|0\right\rangle$$

is quadratically divergent. Here $J_\mu(x)$ is the electromagnetic current and from the current conservation we see that

$$q^\mu \pi_{\mu\nu}(q) = 0, \qquad q^\nu \pi_{\mu\nu}(q) = 0$$

This implies that $\Delta_{\mu\nu}(q)$ has the structure

$$\pi_{\mu\nu}(q) = \left(g_{\mu\nu} q^2 - q_\mu q_\nu\right) \pi\left(q^2\right)$$

Thus only $\Delta(0)$ is logarithmically divergent and counterterm is of the form $\pi(0) F_{\mu\nu} F^{\mu\nu}$. In other words, there is no need for the counter term of the form $A_\mu A^\mu$ as a conquence of the gauge invariance.

As an example, consider the fermion loop contribution to photon self energy, commonly called the vacuum polarization. From the Feynamn rule we have

$$\pi_{\mu\nu}(q) = \int \frac{d^4k}{(2\pi)^4} tr\left(\frac{1}{\not{k} - m}\gamma_\mu \frac{1}{\not{k} + \not{q} - m}\gamma_\nu\right)$$

Then

$$q^\mu \pi_{\mu\nu}(q) = \int \frac{d^4k}{(2\pi)^4} tr\left(\frac{1}{\not{k} - m}\not{q} \frac{1}{\not{k} + \not{q} - m}\gamma_\nu\right)$$

We can write

$$\not{q} = (\not{k} + \not{q} - m) - (\not{k} - m)$$

to get

$$q^\mu \pi_{\mu\nu}(q) = \int \frac{d^4k}{(2\pi)^4} tr[(\frac{1}{\not{k} - m} - \frac{1}{\not{k} + \not{q} - m})\gamma_\nu]$$

If we are allowed to shift the intergration varible $k \to k - q$ in the second term, then we have the cancellation and get the result,

$$q^\mu \pi_{\mu\nu}(q) = 0.$$

But, the integal is highly divergent and the shifting of the integration variable is not allowed. However, if we regulate this integral by dimensional integration, $d^4k \to d^nk$. Then the integral is convergent for sufficiently small n and we can then shift the integration variable to get the result we want. This example illustrate that the relation between vertex function and the propagators, usually referred to as Ward Identity, is indepedent of dimensions and dimensional regulaization is most useful for them.

(b) $B = 1$, $F = 2$, and $D = 0$
Here the vertex function is logarithmically divergent and needs a counterterm of the form $\overline{\psi}(x)\gamma^\mu A_\mu \psi(x)$, which is of the same form as the interaction term in the original Lagrangian.

(c) $B = 4$, $F = 0$, and $D = 0$
This seems to give a logarithmically divergent contribution to the light by light scattering graph and would require a counter term of the form A^4. Again gauge invariance would come in to save the day. To see this we write the light by light scattering amplitude as

$$\begin{aligned} T &= \int \prod_{i=1}^{4} \left[d^4x_i e^{-iq_i \cdot x_i}\right] \varepsilon^\mu(q_1)\,\varepsilon^\nu(q_2)\,\varepsilon^\alpha(q_3)\,\varepsilon^\beta(q_4)\,\langle 0 \left| T\left(J_\mu(x_1) J_\nu(x_2) J_\alpha(x_3) J_\beta(x_4)\right)\right| 0\rangle \\ &= \varepsilon^\mu(q_1)\,\varepsilon^\nu(q_2)\,\varepsilon^\alpha(q_3)\,\varepsilon^\beta(q_4)\, T_{\mu\nu\alpha\beta}(q_1, q_2, q_3, q_4) \end{aligned}$$

It is easy to see from gauge invariance that

$$q_1^\mu T_{\mu\nu\alpha\beta}(q_1, q_2, q_3, q_4) = 0, \qquad q_2^\nu T_{\mu\nu\alpha\beta}(q_1, q_2, q_3, q_4) = 0, \cdots$$

This means that in the Taylor expansion of $T_{\mu\nu\alpha\beta}(q_1, q_2, q_3, q_4)$ the constant term, the term without deivative, vanishes and there is no need for the counterterm.♠

(b) Massive vector field

Here the free Lagrangian is of the form,

$$\mathcal{L}_0 = -\frac{1}{4}\left(\partial_\mu V_\nu - \partial_\nu V_\mu\right)^2 + \frac{1}{2}M_V^2 V_\mu^2$$

where V_μ is a massive vector field and M_V is the mass of the vector field. The propagator in momentum space is of the form,

$$D_{\mu\nu}\left(k\right) = \frac{-i\left(g_{\mu\nu} - k_\mu k_\nu / M_V^2\right)}{k^2 - M_V^2 + i\varepsilon} \longrightarrow O\left(1\right), \quad \text{as} \quad k \to \infty \tag{15}$$

To see this we write the Lagrangian as

$$\int d^4x\mathcal{L}_0 = \int d^4x\frac{1}{2}\left[V_\mu \partial^2 V^\mu - V_\mu \partial^\mu \partial^\nu V_\nu + M_V^2 V_\mu V^\mu\right] = \int d^4x\frac{1}{2}V_\mu[\left(\partial^2 + M_V^2\right)g^{\mu\nu} - \partial^\mu\partial^\nu]V_\nu$$

The propagator is defined by

$$[\left(\partial^2 + M_V^2\right)g^{\mu\nu} - \partial^\mu\partial^\nu]G_{\nu\beta}(x-y) = g_\beta^\mu \delta^4\left(x-y\right)$$

We solve this equation by the Fourier transform,

$$G_{\nu\beta}(x-y) = \int d^4x e^{-ikx} D_{\nu\beta}\left(k\right)$$

we get

$$[\left(-k^2 + M_V^2\right)g^{\mu\nu} + k^\mu k^\nu]D_{\nu\beta} = g_\beta^\mu$$

Use the tensor property of $D_{\nu\beta}$ to write

$$D_{\nu\beta} = a g_{\nu\beta} + b k_\beta k_\nu$$

Then

$$a\left[\left(-k^2 + M_V^2\right)g_\beta^\mu + k^\mu k_\beta\right] + b[\left(-k^2 + M_V^2\right)k^\mu k_\beta + k^2 k^\mu k_\beta] = g_\beta^\mu$$

From these we get

$$a = -\frac{1}{k^2 - M_V^2}, \qquad b = \frac{1}{M_V^2}\frac{1}{k^2 - M_V^2}$$

Or

$$D_{\mu\nu}\left(k\right) = \frac{-i\left(g_{\mu\nu} - k_\mu k_\nu / M_V^2\right)}{k^2 - M_V^2 + i\varepsilon} \quad \longrightarrow \quad O(1) \quad \text{as } k \to \infty$$

This implies that canonical dimension of massive vector field is two rather than one,

$$d(V_\mu) = 2$$

The power counting is now modified with superficial degree of divergence given by

$$D = 4 - B - \frac{3}{2}F - V + \sum_i n_i\left(\Delta_i - 4\right)$$

with

$$\Delta_i = b_i + \frac{3}{2}f_i + 2v_i + d_i$$

Here V is the number of external vector lines, v_i is the number of vector fields in the ith type of vertex and Δ_i is the canonical dimension of the interaction term in $\mathcal{L}$. From the formula for Δ_i we see that the only renormalizable interaction involving massive vector field, $\Delta_i \leq 4$, is of the form, $\phi^2 A_\mu$ and is not Lorentz invariant. Thus there is no nontrivial interaction of the massive vector field which is renormalizabel. However, two important exceptions should be noted;

(a) In a gauge theory with spontaneous symmetry breaking, the gauge boson will acquire mass in such a way to preserve the renormalizability of the theory ([?]).

(b) A theory with a neutral massive vector boson coupled to a conserved current is also renormalizable. Heuristically, we can understand this as follows. The propagator in Eq(15) always appears between conserved currents $J^{\mu}(k)$ and $J^{\nu}(k)$ and the $k_{\mu}k_{\nu}/M_V^2$ term will not contribute because of current conservation, $k^{\mu}J_{\mu}(k) = 0$ or in the coordinate space $\partial^{\mu}J_{\mu}(x) = 0$. Then the power counting is essentially the same as for the massless vector field case.

2.3 Composite operator

In some cases, we need to consider Green's function *involving* composite operator, an operator with more than one fields at same space time point. We will now illustrate this by an example. Consider a simple composite operator of the form $\Omega(x) = \frac{1}{2}\phi^2(x)$ in $\lambda\phi^4$ theory. Green's function with one insertion of Ω is of the form,

$$G_{\Omega}^{(n)}(x; x_1, x_2, x_3, ..., x_n) = \left\langle 0|T(\frac{1}{2}\phi^2(x)\phi(x_1)\phi(x_2)...\phi(x_n))|0\right\rangle$$

In momentum space we have

$$(2\pi)^4\delta^4(p+p_1+p_2+...+p_n)G_{\phi^2}^{(n)}(p; p_1, p_2, p_3, ..., p_n) = \int d^4x\, e^{-ipx}\int\prod_{i=1}^{n} d^4x_i e^{-ip_ix_i}G_{\Omega}^{(n)}(x; x_1, x_2, x_3, ..., x_n)$$

In perturbation theory, we can use Wick's theorem to work out these Green's functions in terms of Feynman diagram. For example, to lowest order in λ the 2-point function with one composite operator $\Omega(x) = \frac{1}{2}\phi^2(x)$ is, after using the Wick's theorem,

$$G_{\phi^2}^{(2)}(x; x_1, x_2) = \frac{1}{2}\left\langle 0|T\{\phi^2(x)\phi(x_1)\phi(x_2)\}|0\right\rangle = i\Delta(x - x_1)i\Delta(x - x_2)$$

or in momentum space

$$G_{\phi^2}^{(2)}(p; p_1, p_2) = i\Delta(p_1)i\Delta(p + p_1)$$

This corresponds to diagram (a) in Fig 15. If we truncate the external propagators, we get

$$\Gamma_{\phi^2}^{(2)}(p, p_1, -p_1 - p) = 1$$

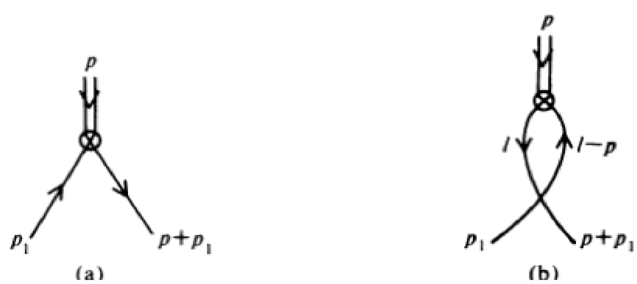


Fig 15 Feynman graphs for composite operator

To first order in λ, we have

$$G_{\phi^2}^{(2)}(x, x_1, x_2) = \int \left\langle 0|T\{\frac{1}{2}\phi^2(x)\phi(x_1)\phi(x_2)\frac{(-i\lambda)}{4!}\phi^4(y)\}|0\right\rangle d^4y$$

$$= \int d^4y\frac{-i\lambda}{2}[i\Delta(x - y)]^2 i\Delta(x_1 - y)i\Delta(x_2 - y)$$

The Feynman diagram is given in diagram (b) of Fig 15. The amputated 1PI momentum space Green's function is

$$\Gamma_{\phi^2}^{(2)}(p; p_1, -p - p_1) = \frac{-i\lambda}{2}\int\frac{d^4l}{(2\pi)^4}\frac{i}{l^2 - \mu^2 + i\epsilon}\frac{i}{(l - p)^2 - \mu^2 + i\epsilon}$$

To calculate this type of Green's functions systematically, we can add a term $\chi(x)\Omega(x)$ to $\mathcal{L}$

$$\mathcal{L}[\chi] = \mathcal{L}[0] + \chi(x)\Omega(x)$$

where $\chi(x)$ is a c-number source function. We can construct the generating functional $W[\chi]$ in the presence of this external source. We obtain the connected Green's function by differentiating $\ln W[\chi]$ with respect to χ and then setting χ. to zero.

Renormalization of composite operators

Superficial drgrees of divergence for Green's function with one composite operator is,

$$D_\Omega = D + \delta_\Omega = D + (d_\Omega - 4)$$

where d_Ω is the canonical dimension of Ω. For the case of $\Omega(x) = \frac{1}{2}\phi^2(x), \quad d_{\phi^2} = 2$ and $D_{\phi^2} = 2 - n \Rightarrow$ only *2-point function* $\Gamma^{(2)}_{\phi^2}$ is divergent. Taylor expansion takes the form,

$$\Gamma^{(2)}_{\phi^2}(p;p_1) = \Gamma^{(2)}_{\phi^2}(0,0) + \Gamma^{(2)}_{\phi^2 R}(p,p_1)$$

We can combine the counter term

$$\frac{-i}{2}\Gamma^{(2)}\phi^2(0,0)\chi(x)\phi^2(x)$$

with the original term to write

$$\frac{-i}{2}\chi\phi - \frac{i}{2}\Gamma^2_{\phi^2}(0,0)\chi\phi^2 = -\frac{i}{2}Z_{\phi^2}\chi\phi^2$$

In general, we need to insert counterterm $\Delta\Omega$ into the original addition

$$L \to L + \chi(\Omega + \Delta\Omega)$$

If $\Delta\Omega = C\Omega$, as in the case of $\Omega = \frac{1}{2}\phi^2$,we have

$$L[\chi] = L[0] + \chi Z_\Omega \Omega = L[0] + \chi\Omega_0$$

with

$$\Omega_0 = Z_\Omega\Omega = (1 + C)\Omega$$

Such composite operators are said to be mutiplicative renormalizable and Green's functions of unrenormalized operator Ω_0 is related to that of renormalized operator Ω by

$$G^{(n)}_{\Omega_0}(x;x_1,x_2,...x_n) = \langle 0|T\{\Omega_0(x)\phi(x_1)\phi(x_2)...\phi(x_n)\}|0\rangle$$

$$= Z_\Omega Z_\phi^{n/2} G^{(n)}_{\Omega R}(x;x_1,...x_n)$$

For more general cases, *the operator in the counter term might not be the operator itself,* $\Delta\Omega \neq c\Omega$ and the renormalization of a composite operator may require counterterm proportional to other composite operators.

Example: Conside 2 composite operators A and B. Denote *their* counterterms by ΔA and ΔB. Including the counter terms with A and B we can write,

$$L[\chi] = L[0] + \chi_A(A + \Delta A) + \chi_B(B + \Delta B)$$

Very often counterterms ΔA and ΔB are linear combinations of A and B

$$\Delta A = C_{AA}A + C_{AB}B$$

$$\Delta B = C_{BA}A + C_{BB}B$$

In this case we say there is operator mixing among A and B. Write

$$L[\chi] = L[0] + (\chi_A\ \chi_B)\{C\}\begin{pmatrix} A \\ B \end{pmatrix} \quad where\ \ \{C\} = \begin{pmatrix} 1+C_{AA} & C_{AB} \\ C_{BA} & 1+C_{BB} \end{pmatrix}$$

Diagonalized $\{C\}$ by bi-unitary transformation

$$U\{C\}V^+ = \begin{pmatrix} Z_{A'} & 0 \\ 0 & Z_{B'} \end{pmatrix}$$

where U, V are 2×2 unitary matrices. Then

$$L[\chi] = L[0] + Z_{A'}\chi_{A'}A' + Z_{B'}\chi_{B'}B'$$

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = V\begin{pmatrix} A \\ B \end{pmatrix} \quad (\chi_{A'}\ \chi_{B'}) = (\chi_A\ \ \chi_B)U$$

We say that the new combinations A' , B' are multiplicatively renormalizable.

3 Renormalization group

Renormalization scheme requires specification of substraction points which introduce new mass scales. As we will see this introduces the concept of energy dependent "coupling constants",

$$e.g\ \ \lambda = \lambda(s)$$

even though the coupling constants in the original Lagranggian are independent of energies.

3.1 Renormalization group equation

In general, there is arbitrariness in choosing the renormalization schemes (or the substraction points). Nevertheless, the physical results should be the same, i.e. independent of renormalization schemes. In essence this is the physical content of the renormalization group equation. Suppose we have different renormalizarion scheme R and R'. From the point of view of BPH renormalization, we can write the bare Lagrangian as, (see Eq(7))

$$\mathcal{L} = \mathcal{L}_R(R - quantities) = \mathcal{L}_{R'}(R' - quantities)$$

Recall that

$$\phi_R = Z_{\varphi R}^{-\frac{1}{2}}\phi_0, \quad \lambda_R = Z_{\lambda R}^{-1}Z_{\phi R}^2\lambda_0 \quad \mu_R^2 = \mu_0^2 + \delta\mu_R^2$$

Similarly,

$$\phi_{R'} = Z_{\varphi R'}^{-\frac{1}{2}}\phi_0, \quad \lambda_{R'} = Z_{\lambda R'}^{-1}Z_{\phi R'}^2\lambda_0 \quad \mu_{R'}^2 = \mu_0^2 + \delta\mu_{R'}^2$$

Since ϕ_0, λ_0 and μ_0 are the same, we can finite relations between $R-$ and R' quantities
Callan-Symanzik equation

Here to conform with the standard notation, we make a change of notation. We will use m_0 and m for bare and renormalized masses instead of μ_0 and μ. The paramter μ is now used to denote the substraction point. In general the renormalized *1PI* Green's functions are related to bare *1PI* Green's by

$$Z_\phi^{-\frac{n}{2}}\Gamma_R^{(n)}(P_i, \lambda, m, \mu) = \Gamma^{(n)}(P_i, \lambda_0, m_0)$$

The renormalized $\Gamma_R^{(n)}(P_i, \lambda, m, \mu)$ depends on the substraction point μ, while the unrenormalied one $\Gamma^{(n)}(P_i, \lambda_0, \mu_0^2)$ does not,

$$\mu\frac{\partial}{\partial\mu}\Gamma^{(n)}(P_i, \lambda_0, \mu_0^2) = 0, \qquad \text{or} \qquad \mu\frac{\partial}{\partial\mu}[Z_\phi^{-\frac{n}{2}}\Gamma_R^{(n)}(P_i, \lambda, \mu)] = 0$$

Using the μ dependence of Z, λ, m we get

$$\left[\mu\frac{\partial}{\partial\mu}+\beta\left(\lambda\right)\frac{\partial}{\partial\lambda}+m\left(\lambda\right)\frac{\partial}{\partial m}-n\gamma\left(\lambda\right)\right]\Gamma_R^{(n)}(P_i,\lambda,\mu)=0$$

where

$$\beta\left(\lambda\right)=\mu\frac{\partial\lambda}{\partial\mu},\qquad m\left(\lambda\right)=\mu\frac{\partial m}{\partial\mu},\qquad \gamma\left(\lambda\right)=\frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_\phi$$

This is usually referred to as **Callan-Symanzik** equation, or renormalization group equation.

Now we want to study the behavior of the Green's functions as we vary the substraction scale μ. In particular, we want to know the asymptotic behavior at high energies. We will use this parameter μ to scale all other dimensional variable like momenta P_i by defining a dimensionless quantity $\bar{\Gamma}$ as

$$\Gamma^{(n)}(P_i,\lambda,m,\mu)=\mu^{4-n}\bar{\Gamma}_R^{(n)}(\frac{P_i}{\mu},m,\lambda)$$

For simplicity, we will study the theory with $m=0$. Since $\bar{\Gamma}$ is dimensionless, as we scale up the momenta P_i,we can write

$$(\mu\frac{\partial}{\partial\mu}+\sigma\frac{\partial}{\partial\sigma})\bar{\Gamma}_R^{(n)}(\frac{\sigma P_i}{\mu},\lambda)=0$$

and

$$[\mu\frac{\partial}{\partial\mu}+\sigma\frac{\partial}{\partial\sigma}+(n-4)]\Gamma^{(n)}(\frac{\sigma P_i}{\mu},\lambda)=0$$

From Callan-Symanzik equation we get

$$[\sigma\frac{\partial}{\partial\sigma}-\beta(\lambda)\frac{\partial}{\partial\lambda}+n\gamma(\lambda)+(n-4)]\Gamma^{(n)}(\sigma p_i,\lambda,\mu)=0$$

To solve this equation, we remove the non-derivative terms by the transformation

$$\Gamma_{as}^{(n)}(\sigma p_i,\lambda,\mu)=\sigma^{4-n}\exp[n\int_0^\lambda\frac{\gamma(x)}{\beta(x)}dx]\Gamma^{(n)}(\sigma p_i,\lambda,\mu)$$

Then $\Gamma^{(n)}$ satisfies the equation

$$[\sigma\frac{\partial}{\partial\sigma}-\beta(\lambda)\frac{\partial}{\partial\lambda}]\Gamma^{(n)}(\sigma p_i,\lambda,\mu)=0$$

or

$$[\frac{\partial}{\partial t}-\beta(\lambda)\frac{\partial}{\partial\lambda}]\Gamma^{(n)}(e^tp_i,\lambda,\mu)=0\quad \text{where}\quad t=\ln\sigma \tag{16}$$

Introduce the effective, or running constant $\bar{\lambda}$ as solution to the equation

$$\frac{d\bar{\lambda}(t,\lambda)}{dt}=\beta(\bar{\lambda})\qquad \text{with initial condition}\quad \bar{\lambda}(0,\lambda)=\lambda$$

This equation has the solution

$$t=\int_\lambda^{\bar{\lambda}(t,\lambda)}\frac{dx}{\beta(x)}$$

It is straightforward to show that

$$\frac{1}{\beta(\bar{\lambda})}\frac{d\bar{\lambda}}{d\lambda}=\frac{1}{\beta(\lambda)}\quad\text{and}\quad[\frac{\partial}{\partial t}-\beta(\lambda)\frac{\partial}{\partial\lambda}]\bar{\lambda}(t,\lambda)=0$$

This means that any funciton $F^{(n)}$ which depends on t and λ through the combination $\bar{\lambda}(t,\lambda)$ will satisfy the renormalization equation in Eq(16)

$$[\frac{\partial}{\partial t}-\beta(\lambda)\frac{\partial}{\partial\lambda}]F^{(n)}(p_i,\bar{\lambda}(t,\lambda),\mu)=0$$

For the other factor of the renormalization group equation, we can write it in terms of effective coupling constant as

$$\begin{aligned}\exp[n\int_0^{\lambda}\frac{\gamma(\lambda)}{\beta(\lambda)}d\lambda] &\sim \exp[n\int_0^{\bar{\lambda}}\frac{\gamma(x)}{\beta(x)}dx + n\int_{\bar{\lambda}}^{\lambda}\frac{\gamma(x)}{\beta(x)}dx] \\ &= H(\bar{\lambda})\exp[-n\int_{\lambda}^{\bar{\lambda}}\frac{\gamma(x)}{\beta(x)}dx]\end{aligned}$$

where

$$H(\bar{\lambda}) = exp[n\int_0^{\bar{\lambda}}\frac{\gamma(x)}{\beta(x)}dx]$$

The solution is then

$$\Gamma_{as}^{(n)}(\sigma p_i, \lambda, \mu) = \sigma^{4-n}\exp[-n\int_0^t \gamma(\bar{\lambda}(x', \lambda))dx']H(\bar{\lambda})F^{(n)}(p_i, \bar{\lambda}(t, \lambda), \mu)$$

If we set $t = 0$ (or $\sigma = 0$), we see that

$$\Gamma_{as}^{(n)}(p_i, \lambda, \mu) = H(\lambda)F^{(n)}(p_i, \lambda, \mu)$$

Thus the solution has the simple form

$$\Gamma_{as}^{(n)}(\sigma p_i, \lambda, \mu) = \sigma^{4-n}exp[-n\int_0^t \gamma(\bar{\lambda}(x', \lambda))dx']\Gamma_{as}^{(n)}(p_i, \bar{\lambda}(t, \lambda), \mu)$$

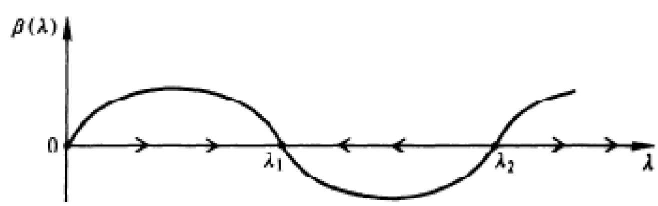
We now discuss the properties of this solution.The first factor σ^{4-n} comes from the naive dimension of the Green's function $\Gamma_{as}^{(n)}(p_i, \bar{\lambda}(t, \lambda), \mu)$ as we scale up the momenta by increasing σ. The second factor gives the dependence on $t = \ln\sigma$ apart from the navie diemension and is usually called the anomalous dimension. The last factor is the original unscaled Green's function but with the coupling constant replaced by the effective coupling constant $\bar{\lambda}(t, \lambda)$.

3.2 Effective coupling constant

We now study the asymptotic behavior of the effective coupling constant $\bar{\lambda}(t, \lambda)$ in order to understand the asymptotic behavior of the Green's functions. The differential equation goverens the behavior of $\bar{\lambda}(t, \lambda)$ is a first order equation and can be studies as follows.

$$\frac{d\bar{\lambda}(t, \lambda)}{dt} = \beta(\bar{\lambda}) \qquad \text{initial condition } \bar{\lambda}(0, \lambda) = \lambda \tag{17}$$

Consider the case where $\beta(\lambda)$ has the following simple behavior



Suppose $0 < \lambda < \lambda_1$, then from Eq(17) we see that,

$$\frac{d\bar{\lambda}}{dt}\,|_{t=0} > 0$$

$\Rightarrow \bar{\lambda}$ increases as t increases. This increase will continue until $\bar{\lambda}$ reaches λ_1, where $\dfrac{d\bar{\lambda}}{dt} = 0$

On the other hand, if initially $\lambda_1 < \lambda < \lambda_2$, then

$$\frac{d\bar{\lambda}}{dt}\,|_{t=0} < 0$$

$\bar{\lambda}$ will decrease until it reaches λ_1. Thus as $t \to \infty$, we get

$$\lim_{t\to\infty} \bar{\lambda}(t,\lambda) = \lambda_1 \qquad \lambda_1 : \text{ultraviolet stable fixed point}$$

and

$$\Gamma^{(n)}_{as}(p_i, \bar{\lambda}(t,\lambda), \mu) \to_{t\to\infty} \Gamma^{(n)}_{as}(p_i, \lambda_1, \mu)$$

Example: Suppose $\beta(x)$ has a simple zero at $\lambda = \lambda_1$,

$$\beta(\lambda) \simeq a(\lambda_1 - \lambda) \qquad a > 0$$

Then

$$\frac{d\bar{\lambda}}{dt} = a(\lambda_1 - \lambda) \Rightarrow \bar{\lambda} = \lambda_1 + (\lambda - \lambda_1)e^{-at}$$

i.e. the approach to fixed point is exponential in t, or power in $t = \ln \sigma$. Also the prefactor can be simplified,

$$\int_0^t \gamma(\bar{\lambda}(x,\lambda))dx = \int_\lambda^{\bar{\lambda}} \frac{\gamma(y)dy}{\beta(y)} \approx \frac{-\gamma(\lambda_1)}{a}\int_\lambda^{\bar{\lambda}} \frac{d\lambda'}{\lambda' - \lambda_1} = \frac{-\gamma(\lambda_1)}{a}\ln(\frac{\bar{\lambda}-\lambda_1}{\lambda-\lambda_1})$$

$$= \gamma(\lambda_1)t = \gamma(\lambda_1)\ln\sigma$$

$$\lim_{\sigma\to\infty} \Gamma^{(n)}_{as}(\sigma p_i, \lambda, \mu) = \sigma^{4-n[1+\gamma(\lambda_1)]}\Gamma^{(n)}_{as}(p_i, \lambda_1, \mu)$$

Thus the asymptotic behavior in field theory is controlled by the fixed point λ_1 and $\gamma(\lambda_1)$ anomalous dimension.

(a) **Asymptotic Freedom**

It is the zeros of the $\beta - function$ of the effective coupling constant $\bar{\lambda}(t,\lambda)$ which determines the aymptotic behavior of the Green's functions. But the locations of the zeros of $\beta - function$ need nonperturbative computation and is beyond reach in most cases. However, pertuabation theory can tell us the behavior of the $\beta - function$ near the origin of the coupling constant space. As we will the discuss later in Chapter on QCD, to explain the data from deep-inelastic electron-proton scattering, it require some theory which behaves like free field theory in the high energy regime. Thie is usually referred to as asymptotic freedom. We now examine the $\beta - function$ for each of the renomalizable theory we have studied.

(a) $\lambda\phi^4$ theory

The Lagrangian is

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu\phi)^2 - m^2\phi^2] - \frac{\lambda}{4!}\phi^4$$

Effective coupling constant $\bar{\lambda}$ *to lowest orde*r satisfies the differential equation

$$\frac{d\bar{\lambda}}{dt} = \beta(\bar{\lambda}) \ , \quad \beta(\lambda) \approx \frac{3\lambda^2}{16\pi^2} + O(\lambda^3)$$

It is not asymptotically free.

The generalization to more than one scalar fields is the replacement,

$$\lambda\phi^4 \to \lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l \ , \qquad \lambda_{ijkl} \text{ is totally symmetric}$$

Then the differential equations are of the form,

$$\beta_{ijkl} = \frac{d\lambda_{ijkl}}{dt} = \frac{1}{16\pi^2}\left[\lambda_{ijmn}\lambda_{mnkl} + \lambda_{ikmn}\lambda_{mnjl} + \lambda_{ilmn}\lambda_{mnjk}\right]$$

For the special case, $i = j = k = l = 1$, we wee that $\beta_{1111} = \frac{3}{16\pi^2}\lambda_{iimn}\lambda_{mn11} > 0$ and theory is not asymptotically free.

(b) Yukawa interaction

Here we need to include the scalar self interaction $\lambda\phi^4$ in order to be renormalizable

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \frac{1}{2}[(\partial_\mu\phi)^2 - \mu^2\phi^2] - \lambda\phi^4 + f\bar{\psi}\psi\phi$$

Now we have a coupled differential equations,

$$\beta_\lambda = \frac{d\lambda}{dt} = A\lambda^2 + B\lambda f^2 + Cf^4, \qquad A > 0$$

$$\beta_f = \frac{df}{dt} = Df^3 + E\lambda^2 f, \qquad D > 0$$

To get $\beta_\lambda < 0$, with $A > 0$, we need $f^2 \sim \lambda$. This means we can drop E term in β_f. With $D > 0$, Yukawa coupling f is not asymptotically free. Generalization to the cases of more than one fermion fields or more scalar fields will not change the situation.

(c) Abelian gauge theory(QED)

The Lagrangian is of the usual form,

$$\mathcal{L} = \bar{\psi}i\gamma^\mu(\partial_\mu - ieA_\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

The effective coupling constant $\bar{e}$ satisfies the equation,

$$\frac{d\bar{e}}{dt} = \beta_e = \frac{\bar{e}^3}{12\pi^2} + O(e^5)$$

For the scalar QED we have

$$\frac{d\bar{e}}{dt} = \beta'_e = \frac{\bar{e}^3}{48\pi^2} + O(e^5)$$

Both are not asymptotically free.

(d) Non-Abelian gauge theories

It turns out that only non-Abelian gauge theories are asymptotically free. Write the Lagrangian as

$$\mathcal{L} = -\frac{1}{2}T_r(F_{\mu\nu}F^{\mu\nu})$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \qquad A_\mu = T_a A^a_\mu$$

and

$$[T_a, T_b] = if_{abc}T_c\ , \quad T_r(T_a, T_b) = \frac{1}{2}\delta_{ab}$$

The evolution of the effective coupling constant is governed by

$$\frac{dg}{dt} = \beta(g) = -\frac{g^3}{16\pi^2}(\frac{11}{3})t_2(V) < 0$$

Since $\beta(g) < 0$ for small g, this theory is asymptotically free. Here

$$t_2(V)\delta^{ab} = t_r[T_a(V)T_b(V)] \qquad t_2(V) = n \quad for \quad SU(n)$$

If gauge fields couple to fermions and scalars with representation matrices, $T^a(F)$ and $T^a(s)$ respectively, then

$$\beta_g = \frac{g^3}{16\pi^2}\left[-\frac{11}{3}t_2(V) + \frac{4}{3}t_2(F) + \frac{1}{3}t_2(s)\right]$$

where

$$t_2(F)\delta^{ab} = t_r(T^a(F)T^b(F))$$
$$t_2(S)\delta^{ab} = t_r(T^a(S)T^b(S))$$

4 Appendix n-dimensional Integration

4.1 n-dimensional "spherical" coordinates

In a n-dimensional space, the Cartesian coordinates can be parametrized in terms of the "spherical" coordinates as

$$\begin{aligned} x_1 &= r_n \sin\theta_{n-1} \sin\theta_{n-2} .... \sin\theta_2 \sin\theta_1, \\ x_2 &= r_n \sin\theta_{n-1} \sin\theta_{n-2} .... \sin\theta_2 \cos\theta_1, \\ x_3 &= r_n \sin\theta_{n-1} \sin\theta_{n-2} .... \sin\theta_3 \cos\theta_2, \\ & .... \\ x_n &= r_n \cos\theta_{n-1} \end{aligned} \tag{18}$$

where

$$0 \le \theta_1 \le 2\pi, \quad 0 \le \theta_2, \theta_3 .... \theta_{n-1} \le \pi, \tag{19}$$

and

$$r_n^2 = x_1^2 + x_2^2 + ... + x_n^2 \tag{20}$$

We want to show that the n-dimensional infinitesimal volume is given by

$$\begin{aligned} & dx_1 dx_2 dx_{3...} dx_n \\ = \; & r_n^{n-1} (\sin\theta_{n-1})^{n-2} (\sin\theta_{n-2})^{n-3} .... (\sin\theta_2) (d\theta_1 d\theta_2 .... d\theta_{n-1}) \, dr_n \end{aligned} \tag{21}$$

We will do this by finding the relation between the volume factors in n and $n-1$ dimensions. Namely, we will proceed from the simplest $n = 2$ to higher and higher dimensional cases:

$$(n = 2) \longrightarrow (n = 3) \longrightarrow (n = 4) \longrightarrow (\text{general } n) \tag{22}$$

4.1.1 (a) $n = 2$

Here the two Cartesian coordinates (x_1, x_2) are related to the familiar polar coordinates (r_2, θ_1),

$$x_1 = r_2 \sin\theta_1, \quad x_2 = r_2 \cos\theta_1, \qquad 0 \le \theta_1 \le 2\pi, \qquad r_2^2 = x_1^2 + x_2^2 \tag{23}$$

The distance ds_2 between neighboring points is given by

$$(ds_2)^2 = (dx_1)^2 + (dx_2)^2 = (dr_2)^2 + r_2^2 (d\theta_1)^2 \tag{24}$$

The "volume" element is then the product of segments in orthogonal directions — *i.e.* the product of the coordinate differential with the appropriate coefficients as indicated by the (quadratic) distance relation:

$$dV_2 = dx_1 dx_2 = (dr_2)(r_2 d\theta_1) = r_2 dr_2 d\theta_1. \tag{25}$$

4.1.2 (b) $n = 3$

Consider a sphere in three-dimensions. If we cut this sphere by a plane perpendicular to x_3-axis, we get a series of circles in the planes spanned by Cartesian coordinates (x_1, x_2) which are related to the polar coordinates (r_2, θ_1)

$$x_1 = r_2 \sin\theta_1, \quad x_2 = r_2 \cos\theta_1, \qquad 0 \le \theta_1 \le 2\pi \qquad r_2^2 = x_1^2 + x_2^2 \tag{26}$$

The infinitesimal distance on this plane can be expressed in these two coordinate systems as

$$(dx_1)^2 + (dx_2)^2 = (dr_2)^2 + r_2^2 (d\theta_1)^2 \tag{27}$$

We can also cut the sphere by a plane containing the x_3 axis, resulting in a series of "vertical circles". On these two dimensional subspaces, the Cartesian coordinates are (r_2, x_3) and the corresponding polar coordinates are $(r_{3,}\theta_2)$. .We recognize that θ_2 is the usual polar angle.

$$\begin{aligned} r_2 &= r_3 \sin\theta_2, \quad x_3 = r_3\cos\theta_2, \qquad 0 \le \theta_2 \le \pi & (28) \\ r_3^2 &= r_2^2 + x_3^2 = x_1^2 + x_2^2 + x_3^2, & (29) \end{aligned}$$

Just like eqn (30), the infinitesimal distance can be expressed in two equivalent ways:

$$(dr_2)^2 + (dx_3)^2 = (dr_3)^2 + r_3^2 (d\theta_2)^2 \tag{30}$$

Combining the two sets of coordinates in eqns (26) and (28), we get the usual spherical coordinate relations,

$$\begin{aligned} x_1 &= r_3 \sin\theta_2 \sin\theta_1 \\ x_2 &= r_3 \sin\theta_2 \cos\theta_1 \\ x_3 &= r_3 \cos\theta_2 \end{aligned} \tag{31}$$

We can turn the distance formula,

$$(ds_3)^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2,$$

into

$$(ds_3)^2 = (dr_2)^2 + r_2^2 (d\theta_1)^2 + (dx_3)^2 \tag{32}$$

by using eqns (27). This can be further reduced, by eqn (30) and (28), into

$$\begin{aligned} (ds_3)^2 &= (dx_1)^2 + (dx_2)^2 + (dx_3)^2 \\ &= (dr_3)^2 + r_2^2 (d\theta_1)^2 + r_3^2 (d\theta_2)^2 \\ &= (dr_3)^2 + r_3^2 \sin\theta_2^2 (d\theta_1)^2 + r_3^2 (d\theta_2)^2 \end{aligned} \tag{33}$$

The volume element can be obtained from the product of these three terms

$$dV_3 = (dr_3)(r_3 \sin\theta_2 d\theta_1)(r_3 d\theta_2) = r_3^2 \sin\theta_2 (dr_3 d\theta_1 d\theta_2). \tag{34}$$

4.1.3 (c) $n = 4$

We can also imagine cutting the sphere in four-dimensions by the manifold ($x_4 =$ constant) to get 3-sphere with radius, $r_3 = r_4 \sin\theta_3$, where we can introduce three-dimensional spherical coordinates, $(r_3, \theta_1, \theta_2)$ as in eqn (31)

$$\begin{aligned} x_1 &= r_3 \sin\theta_2 \sin\theta_1 = (r_4 \sin\theta_3) \sin\theta_2 \sin\theta_1 \\ x_2 &= r_3 \sin\theta_2 \cos\theta_1 = (r_4 \sin\theta_3) \sin\theta_2 \cos\theta_1 \\ x_3 &= r_3 \cos\theta_2 = (r_4 \sin\theta_3) \cos\theta_2 \end{aligned} \tag{35}$$

with the distance formula (33)

$$(dx_1)^2 + (dx_2)^2 + (dx_3)^2 = (dr_3)^2 + r_3^2 (d\theta_2)^2 + r_3^2 \sin^2\theta_2 (d\theta_1)^2 \tag{36}$$

Now we introduce two-dimensional polar coordinates (r_4, θ_3) in the (r_3, x_4) plane

$$r_3 = r_4 \sin\theta_3, \qquad x_4 = r_4 \cos\theta_3 \tag{37}$$

with the distance formula

$$(dr_3)^2 + (dx_4)^2 = (dr_4)^2 + r_4^2 (d\theta_3)^2 \tag{38}$$

In this way the infinitesimal distance in this four dimensional space can be rewritten by using eqns (36) and (38)

$$\begin{aligned}
(ds_4)^2 &= \left[(dx_1)^2 + (dx_2)^2 + (dx_3)^2\right] + (dx_4)^2 \qquad (39)\\
&= \left[(dr_3)^2 + r_3^2 (d\theta_2)^2 + r_3^2 \sin^2\theta_2 (d\theta_1)^2\right] + (dx_4)^2 \\
&= \left[(dr_3)^2 + (dx_4)^2\right] + r_3^2 (d\theta_2)^2 + r_3^2 \sin^2\theta_2 (d\theta_1)^2 \\
&= (dr_4)^2 + r_4^2 (d\theta_3)^2 + r_3^2 (d\theta_2)^2 + r_3^2 \sin^2\theta_2 (d\theta_1)^2
\end{aligned}$$

The infinitesimal volume element is then

$$\begin{aligned}
dv_4 &= (dr_4)(r_4 d\theta_3)(r_3 d\theta_2)(r_3 \sin\theta_2 d\theta_1) \\
&= r_4^3 \sin^2\theta_3 \sin\theta_2 (dr_4 d\theta_1 d\theta_2 d\theta_3) \qquad (40)
\end{aligned}$$

where we have used eqn (37) to reach the last expression.

4.1.4 (d) *General n*

$$\begin{aligned}
x_1 &= r_n \sin\theta_{n-1} \sin\theta_{n-2} .... \sin\theta_2 \sin\theta_1, \\
x_2 &= r_n \sin\theta_{n-1} \sin\theta_{n-2} .... \sin\theta_2 \cos\theta_1, \\
x_3 &= r_n \sin\theta_{n-1} \sin\theta_{n-2} .... \sin\theta_3 \cos\theta_2, \\
& .... \\
x_n &= r_n \cos\theta_{n-1} \qquad (41)
\end{aligned}$$

where

$$r_n^2 = x_1^2 + x_2^2 + ... + x_n^2, \quad 0 \le \theta_1 \le 2\pi, \quad 0 \le \theta_2, \theta_3 ... \theta_{n-1} \le \pi \qquad (42)$$

The infinitesimal distance is

$$\begin{aligned}
(ds_n)^2 &= (dx_1)^2 + (dx_2)^2 + ... + (dx_n)^2 \qquad (43)\\
&= (dr_n)^2 + r_n^2 (d\theta_{n-1})^2 + r_n^2 \sin^2\theta_{n-1} (d\theta_{n-2})^2 \\
&\quad + r_n^2 \sin^2\theta_{n-1} \sin^2\theta_{n-2} (d\theta_{n-3})^2 + ... \\
&\quad + r_n^2 \sin^2\theta_{n-1} \sin^2\theta_{n-2} ...... \sin^2\theta_2 (d\theta_1)^2
\end{aligned}$$

and the volume element is

$$\begin{aligned}
dV_n &= dx_1 dx_2 ..... dx_n \qquad (44)\\
&= (r_n)^{n-1} (\sin\theta_{n-1})^{n-2} (\sin\theta_{n-2})^{n-3} .... \sin\theta_2 (dr_n d\theta_1 d\theta_2 .... d\theta_{n-1})
\end{aligned}$$

4.2 Some integrals in dimensional regularization

Use the dimensional regularization we can derive the following results for the Feynman integrals with denominator power α in n dimensions:

(a)

$$\begin{aligned}
I_0(\alpha, n) &= \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 + 2p \cdot k + M^2 + i\varepsilon)^\alpha} \\
&= i \frac{(-\pi)^{\frac{n}{2}}}{(2\pi)^n} \frac{\Gamma\left(\alpha - \frac{n}{2}\right)}{\Gamma(\alpha)} \frac{1}{(M^2 - p^2 + i\varepsilon)^{\alpha - \frac{n}{2}}} \qquad (45)
\end{aligned}$$

(b)

$$\begin{aligned} I_\mu(\alpha,n) &= \int \frac{d^nk}{(2\pi)^n}\frac{k_\mu}{(k^2+2p\cdot k+M^2+i\varepsilon)^\alpha} \\ &= -p_\mu I_0(\alpha,n) \end{aligned} \tag{46}$$

(c)

$$\begin{aligned} I_{\mu\nu}(\alpha,n) &= \int \frac{d^nk}{(2\pi)^n}\frac{k_\mu k_\nu}{(k^2+2p\cdot k+M^2+i\varepsilon)^\alpha} \\ &= I_0(\alpha,n)\left[p_\mu p_\nu+\frac{1}{2}g_{\mu\nu}\frac{M^2-p^2}{\left(\alpha-\frac{n}{2}-1\right)}\right] \end{aligned} \tag{47}$$

$$\begin{aligned} I_{\mu\nu\rho}(\alpha,n) &= \int \frac{d^nk}{(2\pi)^n}\frac{k_\mu k_\nu k_\rho}{(k^2+2p\cdot k+M^2+i\varepsilon)^\alpha} \\ &= I_0(\alpha,n)\left[p_\mu p_\nu p_\rho+\frac{1}{2}(g_{\mu\nu}p_\rho+g_{\mu\rho}p_\nu+g_{\nu\rho}p_\mu)\frac{M^2-p^2}{\left(\alpha-\frac{n}{2}-1\right)}\right] \end{aligned}$$

We will do them in sequence.

(a) In the Feynman integral

$$I_0(\alpha,n) = \int \frac{d^nk}{(2\pi)^n}\frac{1}{(k^2+2p\cdot k+M^2+i\varepsilon)^\alpha}, \tag{48}$$

the denominator can be written as

$$\begin{aligned} D &= k^2+2p\cdot k+M^2+i\varepsilon = (p+k)^2+\left(M^2-p^2\right)+i\varepsilon \\ &= k'^2+\left(M^2-p^2\right)+i\varepsilon \end{aligned} \tag{49}$$

where $k'=k+p$. For the case $p^2\geq M^2$, we can perform the Wick rotation to get

$$D = -\bar{k}^2+M^2-p^2+i\varepsilon = -(\bar{k}^2+a^2) \tag{50}$$

where $a^2=p^2-M^2-i\epsilon$ and

$$I_0(\alpha,n) = i\int \frac{d^nk}{(2\pi)^n}\frac{1}{(-1)^\alpha}\frac{1}{(k^2+a^2)^\alpha}. \tag{51}$$

As usual [see CL-eqn (2.112)], the n-dimensional angular integration gives,

$$\int d\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \tag{52}$$

Then

$$\begin{aligned} I_0(\alpha,n) &= \frac{i(-1)^\alpha}{(2\pi)^n}\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\int_0^\infty \frac{k^{n-1}dk}{(k^2+a^2)^\alpha} \\ &= \frac{i(-1)^\alpha}{(2\pi)^n}\frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\frac{1}{2}\int_0^\infty \frac{t^{\frac{n}{2}-1}dt}{(t+a^2)^\alpha} \\ &= \frac{i(-1)^\alpha}{(2\pi)^n}\frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\frac{1}{(a^2)^{\alpha-\frac{n}{2}}}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\alpha-\frac{n}{2}\right)}{\Gamma(\alpha)} \\ &= \frac{(-\pi)^{\frac{n}{2}}}{(2\pi)^n}\frac{\Gamma\left(\alpha-\frac{n}{2}\right)}{\Gamma(\alpha)}\frac{1}{(M^2-p^2+i\varepsilon)^{\alpha-\frac{n}{2}}} \end{aligned} \tag{53}$$

One of the most common convergent Feynman integral has $\alpha = 3$,

$$
\begin{aligned}
I_0(3,n) &= \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 + 2p \cdot k + M^2 + i\varepsilon)^3} \\
&= \frac{(-\pi)^{\frac{n}{2}}}{(2\pi)^n} \frac{\Gamma\left(3 - \frac{n}{2}\right)}{\Gamma(3)} \frac{1}{(M^2 - p^2 + i\varepsilon)^{3-\frac{n}{2}}}
\end{aligned} \tag{54}
$$

which gives for $n = 4$,

$$
\begin{aligned}
I_0(3,4) &= \int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k^2 + 2p \cdot k + M^2 + i\varepsilon)^3} \\
&= \frac{i}{32\pi^2} \frac{1}{(M^2 - p^2 + i\varepsilon)}
\end{aligned} \tag{55}
$$

(b)

$$
I_\mu(\alpha,n) = \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu}{(k^2 + 2p \cdot k + M^2 + i\varepsilon)^\alpha} \tag{56}
$$

As before, we set $k' = k + p$. Then

$$
I_\mu(\alpha,n) = \int \frac{d^n k'}{(2\pi)^n} \frac{k'_\mu - p_\mu}{(k^2 + a^2)^\alpha} \tag{57}
$$

and the term linear in k'_μ gives zero because of the symmetric integration. The result is

$$
I_\mu(\alpha,n) = -p_\mu I_0(\alpha,n) \tag{58}
$$

(c)

$$
\begin{aligned}
I_{\mu\nu}(\alpha,n) &= \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu k_\nu}{(k^2 + 2p \cdot k + M^2 + i\varepsilon)^\alpha} \\
&= \int \frac{d^n k}{(2\pi)^n} \frac{\left(k'_\mu - p_\mu\right)\left(k'_\nu - p_\nu\right)}{(k^2 + a^2)^\alpha} \\
&= \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu k_\nu}{(k^2 + a^2)^\alpha} + p_\mu p_\nu \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 + a^2)^\alpha}
\end{aligned} \tag{59}
$$

In the first term we can replace $k_\mu k_\nu \to \frac{1}{n} k^2 g_{\mu\nu}$ to get,

$$
\begin{aligned}
\int \frac{d^n k}{(2\pi)^n} \frac{k_\mu k_\nu}{(k^2 + a^2)^\alpha} &= \frac{g_{\mu\nu}}{n} \int \frac{d^n k}{(2\pi)^n} \frac{k^2}{(k^2 + a^2)^\alpha} \\
&= \frac{g_{\mu\nu}}{n} \int \frac{d^n k}{(2\pi)^n} \frac{k^2 + a^2 - a^2}{(k^2 + a^2)^\alpha} \\
&= \frac{g_{\mu\nu}}{n} \left[I_0(\alpha - 1, n) - a^2 I_0(\alpha, n) \right]
\end{aligned} \tag{60}
$$

Using the identity, $\Gamma(x+1) = x\Gamma(x)$,we get,

$$
I_0(\alpha - 1, n) = \frac{(\alpha - 1) a^2}{\left(\alpha - 1 - \frac{n}{2}\right)} I_0(\alpha, n) \tag{61}
$$

and

$$
I_{\mu\nu}(\alpha,n) = \left[p_\mu p_\nu + \frac{1}{2} g_{\mu\nu} [M^2 - p^2] \frac{1}{\left(\alpha - \frac{n}{2} - 1\right)} \right] I_0(\alpha,n) \tag{62}
$$

For the case $\alpha = 4$, we have

$$
\begin{aligned}
I_{\mu\nu}(\alpha, n) &= \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu k_\nu}{(k^2 + 2p\cdot k + M^2 + i\varepsilon)^4} \\
&= \left[p_\mu p_\nu + \frac{1}{2} g_{\mu\nu} [M^2 - p^2] \frac{1}{\left(3 - \frac{n}{2}\right)} \right] I_0(4, n)
\end{aligned}
\tag{63}
$$

which gives for n=4,

$$
I_{\mu\nu}(4, 4) = \left[p_\mu p_\nu + \frac{1}{2} g_{\mu\nu} [M^2 - p^2] \right] \frac{i}{96\pi^2} \frac{1}{(M^2 - p^2 + i\varepsilon)}
\tag{64}
$$

(d)

$$
\begin{aligned}
I_{\mu\nu\rho}(\alpha, n) &= \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu k_\nu k_\rho}{(k^2 + 2p\cdot k + M^2 + i\varepsilon)^\alpha} \\
&= \int \frac{d^n k}{(2\pi)^n} \frac{\left(k'_\mu - p_\mu\right)\left(k'_\nu - p_\nu\right)\left(k'_\rho - p_\rho\right)}{(k^2 + a^2)^\alpha}
\end{aligned}
\tag{65}
$$

Dropping terms with odd powers of k', we get

$$
\begin{aligned}
I_{\mu\nu\rho}(\alpha, n) &= \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 + a^2)^\alpha} \left[-\left(k_\mu k_\nu p_\rho + k_\mu p_\nu k_\rho + p_\mu k_\nu k_\rho\right) - p_\mu p_\nu p_\rho \right] \\
&= -p_\mu p_\nu p_\rho I_0(\alpha, n) - \left(p_\rho I_{\mu\nu} + p_\mu I_{\nu\rho} + p_\nu I_{\mu\rho}\right) \\
&= I_0(\alpha, n) \left[4 p_\mu p_\nu p_\rho + \frac{1}{2} (g_{\mu\nu} p_\rho + g_{\mu\rho} p_\nu + g_{\nu\rho} p\mu) \frac{M^2 - p^2}{\left(\alpha - \frac{n}{2} - 1\right)} \right]
\end{aligned}
$$