1 Tensor analysis

Rotation (active view)- coordinate axes are fixed and the physical system is undergoing a rotation. Let x'_a, x_b be the components of new and old vectors. Then we have

$$x'_a = \sum_b R_{ab} x_b$$

where R_{ab} are elements of matrix which represents rotation. For example, the matrix for the rotation about $z-$axis is of the form,

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that the relation between x'_a and x_b is linear and homogeneous.

Important properties of transformation:

1. R is an orthogonal matrix,

$$RR^T = R^T R = 1, \qquad or \qquad R_{ab}R_{ac} = \delta_{bc}, \quad R_{ab}R_{cb} = \delta_{ac} \tag{1}$$

This property will ensure that the combination $\vec{x}^2 = x_a x_a$ is invariant under rotations,

$$x'_a x'_a = R_{ac} R_{ab} x_c x_b = x_b x_b$$

This can be generalized to the case of 2 arbitrary vectors, $\vec{A}$, $\vec{B}$ with transformation property,

$$A'_a = R_{ab} A_b, \qquad B'_c = R_{cd} B_d$$

Then $\vec{A} \cdot \vec{B} = A_a B_a$ is invariant under rotation and is usually called **scalar product**.

2. Gradient operators are related by

$$\frac{\partial}{\partial x'_a} = \frac{\partial}{\partial x_c}\frac{\partial x_c}{\partial x'_a}$$

From $x_b = \left(R^{-1}\right)_{ba} x'_a$, we get then

$$\frac{\partial}{\partial x'_a} = \left(R^{-1}\right)_{ca} \frac{\partial}{\partial x_c}$$

Thus gradient operator transforms by $\left(R^{-1}\right)^T$. However, for rotations, R is orthogonal, $\left(R^{-1}\right)^T = R$,

$$\frac{\partial}{\partial x'_a} = R_{ac} \frac{\partial}{\partial x_c}$$

i.e. $\dfrac{\partial}{\partial x_a}$ tranforms the same way as x_a.

1.1 Tensors

Suppose we have two vectors, i.e. they have the transformation properties,

$$A_a \to A'_a = R_{ab} A_b, \qquad B_c \to B'_c = R_{cd} B_d$$

then the product transform as

$$A'_a B'_c = R_{ab} R_{cd} A_b B_d$$

We define the second rank tensors as those objects which have the same transformation properties as the product of 2 vectors, i. e.,

$$T_{ac} \to T'_{ac} = \left(R_{ab} R_{cd}\right) T_{bd}$$

Definition of $n-$th rank tensors (Cartesian tensors)
We can extend the definition of higher rank tensors by the transformation properties of products of many vectors,

$$T_{i_1 i_2 \ldots} \to T'_{i_1 i_2 \ldots i_n} = \left(R_{i_1 j_1}\right)\left(R_{i_2 j_2}\right) \cdots \left(R_{i_n j_n}\right) T_{j_1 j_2 \cdots j_n}$$

Note again that these transformations are linear and homogeneous which implies that

$$if \qquad T_{j_1 j_2 \cdots j_n} = 0, \qquad for \text{ all } j_m$$

then they all zero in other coordinate system.

Tensor operations

1. Mulplication by constants
$$(cT)_{i_1 i_2 \ldots i_n} = cT_{i_1 i_2 \ldots i_n}$$

2. Addition of tensors of same rank
$$(T_1 + T_2)_{i_1 i_2 \ldots i_n} = (T_1)_{i_1 i_2 \ldots i_n} + (T_2)_{i_1 i_2 \ldots i_n}$$
This is because both tensors are multiplied by the same set of matrix elements of the rotation matrix,
$$T_{1 i_1 i_2 \ldots} \to T'_{1 i_1 i_2 \ldots i_n} = (R_{i_1 j_1})(R_{i_2 j_2}) \cdots (R_{i_n j_n}) T_{1 j_1 j_2 \cdots j_n}$$
$$T_{2 i_1 i_2 \ldots} \to T'_{2 i_1 i_2 \ldots i_n} = (R_{i_1 j_1})(R_{i_2 j_2}) \cdots (R_{i_n j_n}) T_{2 j_1 j_2 \cdots j_n}$$
Adding these 2 equations, we get
$$T'_{1 i_1 i_2 \ldots i_n} + T'_{2 i_1 i_2 \ldots i_n} = (R_{i_1 j_1})(R_{i_2 j_2}) \cdots (R_{i_n j_n}) (T_{1 j_1 j_2 \cdots j_n} + T_{2 j_1 j_2 \cdots j_n})$$
From these we see that adding tensors of different rank the rotation matrice will not factoriz to form a new tensor.

3. Multiplication of 2 tensors
$$(ST)_{i_1 i_2 \ldots i_n j_1 j_2 \cdots j_m} = S_{i_1 i_2 \ldots i_n} T_{j_1 j_2 \cdots j_m}$$
This follows from the fact the tensors are like products of vectors and product of tensors mimic products of more vectors.

4. Contraction
$$S_{abc} T_{ae} \to \text{3rd rank tensor}$$
This operation corresponds to making 2 of the tensor indices the same and sum ove. This is identical to taking the scalar product of two vectors. More specifically,if
$$S'_{abc} = R_{aa'} R_{bb'} R_{cc'} S_{a'b'c'}, \qquad T'_{de} = R_{dd'} R_{ee'} T_{d'e'}$$
then
$$S'_{abc} T'_{ae} = R_{ad'} R_{ee'} R_{aa'} R_{bb'} R_{cc'} S_{a'b'c'} T_{d'e'} = R_{ee'} R_{bb'} R_{cc'} S_{d'b'c'} T_{d'e'}$$
where we have used the orthogonality relation, $R_{ad'} R_{aa'} = \delta_{d'a'}$. So this is a 3rd rank tensor.

5. Symmetrization
$$\text{if } T_{ab} \text{ 2nd rank tensor} \Rightarrow \qquad T_{ab} \pm T_{ba} \quad \text{are also 2nd rank tensors}$$
To see this we write
$$T'_{ab} = R_{aa'} R_{bb'} T_{a'b'},$$
Change the indices
$$T'_{ba} = R_{bb'} R_{aa'} T_{b'a'}$$
From these we get
$$T'_{ab} + T'_{ba} = R_{aa'} R_{bb'} (T_{a'b'} + T_{b'a'}), \qquad T'_{ab} - T'_{ba} = R_{aa'} R_{bb'} (T_{a'b'} - T_{b'a'}),$$
This simply means that the permutation of tensor indices commutes with the rotation.

6. Special numerical tensors
$$RR^T = 1, \qquad \Rightarrow R_{ij} R_{kj} = \delta_{ik}, \qquad or \qquad R_{ij} R_{kl} \delta_{jl} = \delta_{ik}$$
This means that δ_{ij} can be treated as 2nk rank tensor. Similarly,
$$(\det R)\, \varepsilon_{abc} = \varepsilon_{ijk} R_{ai} R_{bj} R_{ck}$$
ε_{abc} a 3rd rank tensor. The factor $(\det R)$ will distinguish proper ratations from improper rotations. For example the combination
$$\varepsilon_{abc} x_b p_c$$
transform as a vector under proper rotation.

Useful identities for ε_{abc}

$$\varepsilon_{ijk}\,\varepsilon_{ijl} = 2\delta_{kl}, \qquad \varepsilon_{ijk}\,\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

More general notation for tensor transformation (Jackson),

$$x'_a = R_{ab}x_b, \qquad \Rightarrow R_{ab} = \frac{\partial x'_a}{\partial x_b}$$

Then we can wrtie

$$x'_a = \frac{\partial x'_a}{\partial x_b}x_b$$