# Quantum Field Theory
# Chapter 3-Canonical Quantization

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# 1 Appendix 1 local symmetries

## 1.1 Ablian local symmtry

Since the local symmetry now plays a very important role in the formulation of theories of fundamental interactions. Here we give a historical origin of the local symmetry.

The free Maxwll's equations are

$$\overrightarrow{\nabla}\cdot\overrightarrow{B}=0,\quad \overrightarrow{\nabla}\times\overrightarrow{E}+\frac{\partial\overrightarrow{B}}{\partial t}=0,$$

$$\overrightarrow{B}=\nabla\times\overrightarrow{A},\qquad \overrightarrow{E}=-\nabla\phi-\frac{\partial\overrightarrow{A}}{\partial t}$$

We can solve the first two equations by introducibg vector and scalar potentials $\overrightarrow{A},\phi$ as

$$\overrightarrow{B}=\nabla\times\overrightarrow{A},\qquad \overrightarrow{E}=-\nabla\phi-\frac{\partial\overrightarrow{A}}{\partial t} \tag{1}$$

It is convenient to write these relations as

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu\quad\text{with}\quad F^{0i}=\partial^0A^i-\partial^iA^0=-E^i,\quad F^{ij}=\partial^iA^j-\partial^jA^i=-\epsilon_{ijk}B_k$$

For a charge particle moving in the electromagnetic field, the force in the equation of motion is the Lorentz force,

$$m\frac{d^2\overrightarrow{x}}{dt^2}=e\left(\overrightarrow{E}+\overrightarrow{v}\times\overrightarrow{B}\right)$$

The Lagrangian for these equations is

$$L=\frac{1}{2}m\left(\overrightarrow{v}\right)^2+e\overrightarrow{A}\cdot\overrightarrow{v}-eA_0$$

To see this we compute the derivatives with respect to $\overrightarrow{x}$ and $\overrightarrow{v}$,

$$\frac{\partial L}{\partial v_i}=mv_i+eA_i,\quad \frac{\partial L}{\partial x_i}=e\frac{\partial A_j}{\partial x_i}v_j-e\frac{\partial A_0}{\partial x_i}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right)=m\frac{dv_i}{dt}+e\frac{\partial A_i}{\partial x_j}\frac{dx_j}{dt}+e\frac{\partial A_i}{\partial t}$$

Thus the Euler-Lagrange equation gives

$$m\frac{dv_i}{dt}+e\frac{\partial A_i}{\partial x_j}\frac{dx_j}{dt}+e\frac{\partial A_i}{\partial t}=e\frac{\partial A_j}{\partial x_i}v_j-e\frac{\partial A_0}{\partial x_i}$$

On the other hand,

$$\left(\vec{v}\times\vec{B}\right)_i=\varepsilon_{ijk}v_jB_k=\varepsilon_{ijk}v_j\varepsilon_{klm}\partial_lA_m=v_j\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)\partial_lA_m=v_j\left(\partial_iA_j-\partial_jA_i\right)$$

Then we get

$$m\frac{dv_i}{dt}=-e\frac{\partial A_i}{\partial x_j}v_j-e\frac{\partial A_i}{\partial t}+e\frac{\partial A_j}{\partial x_i}v_j-e\frac{\partial A_0}{\partial x_i}$$

Or

$$m\frac{dv_i}{dt}=e\left(\partial_iA_j-\partial_jA_i\right)v_j+e\left(-\partial_iA_0-\partial_0A_i\right)=e\left(\vec{E}+\vec{v}\times\vec{B}\right)_i$$

which is the correct equation of motion.

From the Lagrangian we define the conjugate momentum,

$$p_i=\frac{\partial L}{\partial v_i}=mv_i+eA_i,\qquad\Longrightarrow\qquad v_i=\frac{1}{m}\left(p_i-eA_i\right)$$

Note that this is one of the examples where the conjugate momenta is not simply $m\vec{p}$. The Hamiltonian is then

$$\begin{aligned}H&=p_iv_i-L=p_iv_i-\frac{1}{2}m\left(\vec{v}\right)^2-e\vec{A}\cdot\vec{v}+eA_0\\&=\frac{1}{2m}\left(\vec{p}-e\vec{A}\right)^2+eA_0\end{aligned}$$

Note that we can obain this Hamitonian from the free Hamiltonian $H=\vec{p}^{\,2}/2m$ by the substition,

$$\vec{p}\longrightarrow\vec{p}-e\vec{A},\qquad H\longrightarrow H-eA_0$$

Or

$$p^\mu\longrightarrow p^\mu-eA^\mu$$

This is usally called the **principle of minimal substitution.**

The Schrodinger equation for a charged particle moving in the electromagnetic field is of the form,

$$[-\frac{1}{2m}\left(\vec{\nabla}-ie\vec{A}\right)^2+eA_0]\psi=i\frac{\partial\psi}{\partial t}$$

This shows that it is the potentials $\vec{A}$, $A_0$,not the $\vec{E}$, $\vec{B}$ fields show up in the Schrodinger equation. However, Schrodinger equation is not invariant under the gauge transformation,

$$A^\mu\longrightarrow A^\mu+\partial^\mu\alpha,\qquad\text{or}\qquad\vec{A}\longrightarrow\vec{A}-\vec{\nabla}\alpha,\qquad A_0\longrightarrow A_0+\partial_0\alpha$$

But it turns out that we can recover the Schrodinger equation if we also change the wave function $\psi$ by a phase,

$$\psi\longrightarrow\psi'=e^{-ie\alpha}\psi$$

This can be seen as follows. Define the covariant derivative as

$$\vec{D}\psi=\left(\vec{\partial}-ie\vec{A}\right)\psi$$

The covariant derivative for the new field is then,

$$\begin{aligned}\vec{D}\psi' &= \left(\vec{\partial} - ie\vec{A}'\right)\psi' = e^{-ie\alpha}[\vec{\partial} - ie\vec{\nabla}\alpha - ie\left(\vec{A} - \vec{\nabla}\alpha\right)]\psi \\ &= e^{-ie\alpha}\left(\vec{D}\psi\right)\end{aligned}$$

So the covariant derivative $\vec{D}\psi$ transforms by a phase in the same way as the field $\psi$.In other words, the covariant derivative $\vec{D} = \left(\vec{\partial} - ie\vec{A}\right)$ does not change the transformation property of the object it acts on. It is then easy to see that

$$\vec{D}^2\psi' = e^{-ie\alpha}\left(\vec{D}^2\psi\right)$$

For the time derivative, we have

$$D_0\psi = (\partial_0 + ieA_0)\psi$$

and

$$D_0\psi' = e^{-ie\alpha}(\partial_0 + ie\partial_0\alpha - ieA_0 - ie\partial_0\alpha)\psi = e^{-ie\alpha}D_0\psi$$

With this phase transformation, the Schrodinger equation

$$[-\frac{1}{2m}\left(\vec{\nabla} - ie\vec{A}'\right)^2 + eA_0']\psi' = i\frac{\partial\psi'}{\partial t}$$

becomes

$$e^{-ie\alpha}[-\frac{1}{2m}\left(\vec{\nabla} - ie\vec{A}\right)^2 + eA_0]\psi = e^{-ie\alpha}i\frac{\partial\psi}{\partial t}$$

After cancelling out the phase $e^{-ie\alpha}$, we get back the original Schrodinger equation. The phase transformation of the wave function is a symmetry transformation and is a local symmetry because the phase is a funciton of space-time coordinates, $\alpha = \alpha\left(\vec{x}, t\right)$. The phase transformation in usually referred to as $U(1)$ transformation and we call the elecromagnetic possesses $U(1)$ local symmetry.

### 1.2 Non-Abelian symmetry-Yang Mills fields

In 1954, C. N.Yang and R. Mills generalized the Abelian $U(1)$ local symmetry in the Maxwell theory to the non-Abelian $SU(2)$ local symmetry for the isospin and obtained a theory which is qualitatively different from the Abelian case. To illustrate this we consider a $SU(2)$ doublet $\psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}$. Under SU(2) transformation, we have

$$\psi(x) \rightarrow \psi'(x) = exp\{-\frac{i\vec{\tau}\cdot\vec{\theta}}{2}\}\psi(x)$$

where $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are Pauli matrices,

$$\sigma_1 = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$$

with

$$[\frac{\tau_i}{2}, \frac{\tau_j}{2}] = i\epsilon_{ijk}(\frac{\tau_k}{2})$$

Start from free Lagrangian

$$\mathcal{L}_0 = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi$$

which is invariant under global $SU(2)$ transformation where $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ are independent of $x_\mu$.For local symmetry transformation, write

$$\psi(x) \to \psi'(x) = U(\theta)\psi(x) \qquad U(\theta) = exp\{-\frac{i\vec{\tau}\cdot\theta(\vec{x})}{2}\}$$

Again the derivative term

$$\partial_\mu\psi(x) \to \partial_\mu\psi'(x) = U\partial_\mu\psi + (\partial_\mu U)\psi$$

does not have a simple transformation. Introduce gauge fields $\vec{A_\mu}$ to form the covariant derivative,

$$D_\mu\psi(x) \equiv (\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A_\mu}}{2})\psi$$

and require that it has the same transformation as $\psi$

$$[D_\mu\psi]' = U[D_\mu\psi]$$

This determines the transformation property of the gauge field $\vec{A_\mu}$ ,

$$(\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A_\mu}'}{2})(U\psi) = U(\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A_\mu}}{2})\psi$$

and gives the transformation of gauge fields,

$$\boxed{\frac{\vec{\tau}\cdot\vec{A_\mu}'}{2} = U(\frac{\vec{\tau}\cdot\vec{A_\mu}}{2})U^-1 - \frac{i}{g}(\partial_\mu U)U^-1}$$

We use covariant derivatives to construct field tensor. The term with 2 covariant derivatives can be written as,

$$D_\mu D_\nu\psi = (\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A_\mu}}{2})(\partial_\nu - ig\frac{\vec{\tau}\cdot\vec{A_\nu}}{2})\psi = \partial_\mu\partial_\nu\psi - ig(\frac{\vec{\tau}\cdot\vec{A_\mu}}{2}\partial_\nu\psi + \frac{\vec{\tau}\cdot\vec{A_\nu}}{2}\partial_\mu\psi)$$

$$-ig\partial_\mu(\frac{\vec{\tau}\cdot\vec{A_\nu}}{2})\psi + (-ig)^2(\frac{\vec{\tau}\cdot\vec{A_\mu}}{2})(\frac{\vec{\tau}\cdot\vec{A_\nu}}{2})\psi$$

Antisymmetrize this to get the field tensor,

$$(D_\mu D_\nu - D_\nu D_\mu)\psi \equiv ig(\frac{\vec{\tau}\cdot\vec{F_{\mu\nu}}}{2})\psi$$

then

$$\frac{\vec{\tau}\cdot\vec{F_{\mu\nu}}}{2} = \frac{\vec{\tau}}{2}\cdot(\partial_\mu\vec{A_\nu} - \partial_\nu\vec{A_\mu}) - ig[\frac{\vec{\tau}\cdot\vec{A_\mu}}{2}, \frac{\vec{\tau}\cdot\vec{A_\nu}}{2}]$$

Or in terms of components,

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk}A^i_\mu A^k_\nu$$

The the term quadratic in $A$ is new in Non-Abelian symmetry. Under the gauge transformation we have

$$\vec{\tau}\cdot\vec{F_{\mu\nu}}' = U(\vec{\tau}\cdot\vec{F_{\mu\nu}})U^{-1}$$

Infinitesmal transformation

Sometime we will use the gauge transformations in the infinitesmal form. Then we have for $\theta(x) \ll 1$, the following transformaitons,

$$A^{i\mu} = A^\mu + \epsilon^{ijk}\theta^j A^k_\mu - \frac{1}{g}\partial_\mu\theta^i$$

$$F^i_{\mu\nu} = F^i_{\mu\nu} + \epsilon^{ijk}\theta^j F^k_{\mu\nu}$$

The complete Lagragian for the non-Abelian local symmetry is then

$$\mathcal{L} = -\frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu} + \bar{\psi}(x)(i\gamma^\mu D_\mu - m)\psi$$

where

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk}A^i_\mu A^k_\nu, \qquad D_\mu\psi \equiv (\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A_\mu}}{2})\psi$$

This Lagragian is invariant under the local symmetry transformation

$$A^{i/}_\mu = A^i_\mu + \epsilon^{ijk}\theta^j A^k_\mu - \frac{1}{g}\partial_\mu\theta^i, \qquad \psi(x) \to \psi'(x) = \exp\{-\frac{i\vec{\tau}\cdot\vec{\theta}}{2}\}\psi(x) \tag{2}$$

Remarks:

(a) Again $A^a_\mu A^{a\mu}$ is not gauge invariant and gauge boson are massless which lead to long range force. This is phenomenologically not viable because, there are no other long-range forces besides QED.

(b) Unlike photon which does not carry electric charge, the gauge boson here $A^a_\mu$ carries the symmetry charge, the $SU(2)$ charge.

(c) The quadratic term in field tensor $F^{a\mu\nu} \sim \partial A - \partial A + gAA$ is present only in the non-Abelian symmetry. This feature has led to interesting property,.e.g. asymptotic freedom.

# 2 Appendix 2 Non-relativistic Field Theory

We now discuss the quantum field theory for non-relativistic system. Conceptually they are similar to the relativistic case. But the physical interpretation is somewhat different as we will see.

We first consider the simple case of 1-dimensional Schrodinger equation given by

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x,t) = i\hbar\frac{\partial\psi}{\partial t}$$

The Lagrangian density in this case is

$$\mathcal{L} = -\frac{\hbar^2}{2m}\frac{\partial\psi^\dagger}{\partial x}\frac{\partial\psi}{\partial x} + \psi^\dagger V(x)\psi + i\hbar\psi^\dagger\frac{\partial\psi}{\partial t}$$

We now verify that this does give Schrodinger equation,

$$\frac{\partial\mathcal{L}}{\partial\psi^\dagger} = i\hbar\frac{\partial\psi}{\partial t} - V(x)\psi, \qquad \frac{\partial\mathcal{L}}{\partial(\partial_0\psi^\dagger)} = 0,$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_x\psi^\dagger)} = -\frac{\hbar^2}{2m}\frac{\partial\psi}{\partial x},$$

Euler Lagrange equation of motion

$$\partial_x\frac{\partial\mathcal{L}}{\partial(\partial_x\psi^\dagger)} + \partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0\psi^\dagger)} = \frac{\partial\mathcal{L}}{\partial\psi^\dagger},$$

gives

$$i\hbar\frac{\partial\psi}{\partial t} - V(x)\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

This is the Schrodinger equation.

From the Lagrangain density we get Conjugate momenta

$$\pi(x,t) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\psi\dagger$$

and Hamiltonian density

$$H = \pi\partial_0\psi - \mathcal{L} = i\hbar\psi\dagger\frac{\partial\psi}{\partial t} - [i\hbar\psi\dagger\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\frac{\partial\psi^\dagger}{\partial x}\frac{\partial\psi}{\partial x} + \psi^\dagger V(x)\psi]$$
$$= \frac{\hbar^2}{2m}\frac{\partial\psi^\dagger}{\partial x}\frac{\partial\psi}{\partial x} + \psi^\dagger V(x)\psi$$

For the quantization we impose the commutation relations

$$[\psi(x,t),\ \pi(x',t)] = i\delta(x-x'), \qquad \Longrightarrow \qquad \left[\psi(x,t),\ \psi^\dagger(x,t)\right] = \delta(x-x')$$

Suppose $\phi_n$ are the normalized eigenstates of $H$ for a given $V(x)$,

$$[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)]\phi_n = E_n\phi_n$$

and

$$\int dx\phi_n^*(x)\phi_m(x) = \delta_{nm}$$

Here $E_n, n = 0, 1, 2, \cdots$ are the energies for the eigenstates $\phi_0, \phi_1, \phi_2, \cdots$ .

Mode expansion

We now expand the field operator in terms of these eigenfuntions to introduce the creation and annihilation operators,

$$\psi(x,t) = \sum_n a_n\phi_n(x)e^{-iE_nt}, \qquad \Longrightarrow \qquad \psi(x,t) = \sum_n a_n^\dagger\phi_n^*(x)e^{iE_nt}$$

where $a_n$ and $a_n^\dagger$ are operators. We can invert these relations to get

$$a_n = \int e^{iE_nt}\phi_n^*(x)\psi(x,t), \qquad a_n^\dagger = \int e^{-iE_nt}\phi_m(x)\psi^\dagger(x,t)$$

Note that $a_n$ and $a_n^\dagger$ are time independent. We can compute their commutation relation,

$$\left[a_n,\ a_m^\dagger\right] = e^{iE_nt}e^{-iE_mt}\int dxdx'\left[\psi(x,t),\ \psi^\dagger(x,t)\right]\phi_n^*(x)\phi_m^*(x)$$
$$= e^{iE_nt}e^{-iE_mt}\int dxdx'\delta(x-x')\phi_n^*(x)\phi_m^*(x) = \delta_{nm}$$

Similarly,

$$[a_n,\ a_m] = 0$$

We can write the Hamiltonian as

$$H = \int\left[\frac{\hbar^2}{2m}\frac{\partial\psi^\dagger}{\partial x}\frac{\partial\psi}{\partial x} + \psi^\dagger V(x)\psi\right]dx = \sum_{n,m}[\frac{\hbar^2}{2m}\partial_x\phi_n^*\partial_x\phi_m a_n^\dagger a_m + V(x)\phi_n^*\phi_m a_n^\dagger a_m$$
$$= \sum_{n,m}\int dx[\phi_n^*\left(-\frac{1}{2m}\partial_x^2 + V(x)\right)\phi_m a_n^\dagger a_m = \sum_{n,m}\int dx[\phi_n^* E_m\phi_m a_n^\dagger a_m]$$

Or

$$H = \sum_n E_n a_n^\dagger a_n$$

We see that the Hamiltonian is made out many quanta each with energy $E_{n.}$. Eigenstates of $H$ are

$$|0\rangle\,, \qquad a_n^\dagger\,|0\rangle\,, \qquad a_n^\dagger a_m^\dagger\,|0\rangle\,, \cdots$$

where

$$a_n\,|0\rangle = 0, \qquad \text{for all n}$$

Eigenvalues are

$$E\left(n_1, n_2, \cdots\right) = \sum_k E_k n_k, \qquad n_k, \#\text{ of particles in level } n$$

We see that the Hamiltonian is made out many quanta each with energy $E_{n.}$ and $a_n$. Even though this describes a system with many particles each can occupy an energy level $n$, there is no interaction among different particles due to the feature that the Lagrangian is quadratic in the field operator $\psi$, just like the free fields int relativistic theory. The only interaction is the interaction of each particle with fixed potentials. For example if $V(x) = \frac{1}{2}m\omega^2 x$ is that of a simple harmonic oscillator, then this Hamiltonian describes many particles each interacts with the Hamonic oscillator potential but no interactions among different particles.

It is easy to see that the formalism can be generalized to particles moving in 3-dimensions. One of the important examples is that of the Coulomb potential $V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r}$ and there are many electrons moving this potential due to the nucleus and there is no interaction between electrons.

It is clear that to have interactions between particles we need to introduce in the Lagrangian terms higher than the quadratic in the fields. For example,

$$\int dx\mathcal{L}_{int} = \int d^3x d^3y\ \psi^\dagger(x)\,\psi^\dagger(y)\,V(x-y)\,\psi(x)\,\psi(y)$$

which describes 2 particles interacting through a translaitonal invariant potential $V(x-y)$. Note that this form conserves the particle number which is an important feature of non-relativistic system. This form will be used later in the discussion of theory for the superfluid.

# Summary

Canonical quantization is carried out using the Lagrangian formalism for the scalar, fermion, and electromagnetic fields. The particle interpretation of these fields follows from the structure of the energy and momentum operators. For the fermion anti-commutator quatization is used instead of commutator because of the Fermi-Dirac statistics. The normal ordering is introduced to remove the infinite constant in the vacuum energy. The cases with symmetry are discussed in the context of Noether's theorem. The quantization of the electromagnetic field is performed in the radiation gauge which picks out the transverse polarization of the photon field. In the appendix 1, the quantization of simple harmonic oscillator, which forms the basis of the field theory framework is reviewed. In the appendix 2, the quantization of the non-relativistic field theory is briefly discussed.