QFT-Canonical Quantization

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Chapter 3 Canonical Quantization

Quantization of Free Fields

The quantization of field is a generalization of the quantization in the non-relativistic quantum mechanics where we impose the commutaiton relations for coordinates $q_i, i = 1, 2\cdots, n$ and their conjugate momenta p_j,

$$[q_i, p_j] = i\delta_{ij}$$

where p_j is defined by

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \qquad L : \text{Lagrangian}$$

The Hamiltonian is

$$H = \sum_i p_i \dot{q}_i - L$$

The dynamics is determined by the Schrodinger equation,

$$H\Psi = i\frac{\partial \Psi}{\partial t}$$

Here wave function $\Psi(t)$ gives time evolution while operators $p_{i,} q_j$ are time independent.Thisρ is known as the **Schrodinger picture.** Alternatively, we can go to **Heisenberg picture** where

$p_i(t)$ and $q_j(t)$ carry the time dependence instead of state vector Ψ.This is known as the Hsienberg picture which is related to the Schrodinger picture by unitary transformation,

$$\Psi_S(t) = e^{-iHt}\Psi_H$$

and

$$O_H(t) = e^{iHt}O_S e^{-iHt}$$

In this picture the canonical commutation relation is then

$$[q_i(t), p_j(t)] = i\delta_{ij}$$

In relativistic field theory we will use Heisenberg picture so that both spatial coordinate $\vec{x}$ and t both appear as arguments of the field operator $\phi\left(\vec{x}, t\right)$

Thus in field theory we replace $q_i(t)$ by $\phi\left(\vec{x}, t\right)$. To make this correspondence more transparent, divide the 3-dim space into cells of volume ΔV_i and define the ith coordinate $\phi_i(t)$ by averaging $\phi\left(\vec{x}, t\right)$ over the ith cell

$$\phi_i(t) = \frac{1}{\Delta V_i}\int_{\Delta V_i} d^3x\phi\left(\vec{x}, t\right)$$

Similarly, $\partial_0\phi_i(t)$ is the averge of $\partial\phi\left(\overrightarrow{x},t\right)/\partial t$ over the $i$the cell. Write the Lagrangian L as integration of Lagrangain density $\mathcal{L}$,

$$L = \int d^3x\ \mathcal{L}$$

and let $\mathcal{L}_i$ be the average of $\mathcal{L}$ in the $i$the cell. We define the conjugate momenta as

$$p_i(t) = \frac{\partial L}{\partial\left(\partial_0\ \phi_i(t)\right)} = \Delta V_i \frac{\partial\mathcal{L}_i}{\partial\left(\partial_0\ \phi_i(t)\right)} \equiv \Delta V_i \pi_i(t)$$

The Hamiltonian is then defined as

$$H = \sum_i p_i(t)\,\partial_0\ \phi_i(t) - L = \sum_i \Delta V_i\left(\pi_i\partial_0\ \phi_i(t) - \mathcal{L}_i\right) \longrightarrow \int d^3x\mathcal{H}$$

and

$$\mathcal{H} = \pi_i\partial_0\ \phi_i(t) - \mathcal{L}$$

Canonical commutation relations are

$$\left[\phi_i(t), p_j(t)\right] = i\delta_{ij}, \qquad \left[\phi_i(t), \phi_j(t)\right] = 0, \qquad \left[p_i(t), p_j(t)\right] = 0$$

Or in terms of π_i

$$\left[\phi_i(t),\ \pi_j(t)\right] = i\frac{\delta_{ij}}{\Delta V_i}$$

These become in the continuum language,

$$\left[\phi\left(\vec{x},t\right),\pi\left(\vec{x}',t\right)\right]=i\delta^3\left(\vec{x}-\vec{x}'\right),\qquad\left[\phi\left(\vec{x},t\right),\phi\left(\vec{x}',t\right)\right]=0,$$

$$\left[\pi\left(\vec{x},t\right),\pi\left(\vec{x}',t\right)\right]=0$$

where the Dirac delta function emerges as the limit of $\dfrac{\delta_{ij}}{\Delta V_i}$ as $\Delta V_i\longrightarrow 0$, according to

$$\int d^3x'\delta^3\left(\vec{x}-\vec{x}'\right)f(\vec{x'})=f(\vec{x})$$

Also we have

$$\pi\left(\vec{x},t\right)=\frac{\partial\mathcal{L}}{\partial\left(\partial_0\phi\right)}$$

Scalar field

Consider a scalar field ϕ satifies the Klein-Gordon equation

$$\left(\partial^\mu\partial_\mu+\mu^2\right)\phi=0$$

Lagrangian density is

$$\mathcal{L}=\frac{1}{2}\left(\partial^\mu\phi\right)\left(\partial_\mu\phi\right)-\frac{\mu^2}{2}\phi^2$$

Euler-Lagrange equation for this $\mathcal{L}$

$$\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial^\mu \phi \right)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

gives the Klein-Gordon equation.

$$\partial^\mu \partial_\mu \phi + \mu^2 \phi = 0$$

Canonical quantization

Conjugate momentum

$$\pi\left(\overrightarrow{x},t\right)=\frac{\partial\mathcal{L}}{\partial\left(\partial_0\phi\right)}=\left(\partial_0\phi\right)$$

Impose commutation relations,

$$\left[\phi\left(\overrightarrow{x},t\right),\pi\left(\overrightarrow{y},t\right)\right]=i\delta^3\left(\overrightarrow{x}-\overrightarrow{y}\right),\qquad\left[\phi\left(\overrightarrow{x},t\right),\phi\left(\overrightarrow{y},t\right)\right]=0,\tag{1}$$

$$\left[\pi\left(\overrightarrow{x},t\right),\pi\left(\overrightarrow{y},t\right)\right]=0$$

Hamiltonian density is

$$\mathcal{H}=\pi\partial_0\phi-\mathcal{L}=\frac{1}{2}\left[\left(\partial^0\phi\right)^2+\left(\vec{\nabla}\phi\right)^2\right]+\frac{1}{2}\mu^2\phi^2$$

We can compute the commutator

$$\left[H,\phi(\vec{x},t)\right]=\int d^3y\left[\mathcal{H},\phi(\vec{x},t)\right]=-i\partial_0\phi$$

Thus Hamiltonian generates the time translation.

Mode expansion

To find physical contents, expand in classical solutions,

$$\phi(\overrightarrow{x}, t) = \int \frac{d^3 k}{\sqrt{(2\pi)^3 2w_k}} \left[a(\overrightarrow{k}) e^{-ik\cdot x} + a^\dagger(\overrightarrow{k}) e^{ik\cdot x} \right], \qquad k_0 = \sqrt{\overrightarrow{k}^2 + \mu^2}$$

$a(k)$ and $a^\dagger(k)$ are operators.Note that term $a^\dagger(\overrightarrow{k})e^{ik\cdot x}$ corresponds to the negative energy solution. This will become the creation operator while the first term $a(\overrightarrow{k})e^{-ikx}$ correspond to destruction operator.

Solve $a(k)$ and $a^\dagger(k)$ in ϕ and $\partial_0\phi$.This can be carried out as follows. The derivative of ϕ is

$$\partial_0\phi\left(\vec{x},t\right)=\int\frac{d^3k}{\sqrt{(2\pi)^3\,2w_k}}\left(-ik_0\right)\left[a(\vec{k})e^{-ik\cdot x}-a^\dagger(\vec{k})e^{ik\cdot x}\right],\qquad k_0=\sqrt{\vec{k}^{\,2}+\mu^2}=w_k$$

Combining these two relations and integrating over x after multiplying $e^{ik'x}$, we get

$$\int e^{ik'x}d^3x\left(\partial_0\phi-ik_0\phi\right)=\int\frac{d^3k}{\sqrt{(2\pi)^3\,2w_k}}\left(-2ik_0\right)\delta^3\left(k-k'\right)a\left(k\right)$$

From this we get

$$a(k)=i\int d^3x\frac{1}{\sqrt{(2\pi)^3\,2w_k}}\left[e^{ikx}\partial_0\phi-\left(\partial_0e^{ik\cdot x}\right)\right]$$

If we introduce the notation

$$f\overleftrightarrow{\partial_0}g\equiv f\partial_0g-\left(\partial_0f\right)g$$

we can write

$$a(k)=i\int d^3x\frac{e^{ik\cdot x}}{\sqrt{(2\pi)^3\,2w_k}}\overleftrightarrow{\partial_0}\phi\left(x\right)$$

Hermitian conjugate

$$a^{\dagger}(k) = -i \int \frac{d^3x \; e^{-ik\cdot x}}{\sqrt{(2\pi)^3 \, 2w_k}} \overleftrightarrow{\partial_0} \, \phi \, (x)$$

where

$$f \overleftrightarrow{\partial_0} g \equiv f \partial_0 g - (\partial_0 f) \, g$$

Commutators can be calculated as

$$\left[a(\overrightarrow{k}), a^{\dagger}(\overrightarrow{k'}) \right] = \delta^3(\overrightarrow{k} - \overrightarrow{k'}) \; , \qquad \left[a(\overrightarrow{k}), a(\overrightarrow{k}') \right] = 0$$

For example,

$$\begin{aligned} \left[a(\overrightarrow{k}), \; a^{\dagger}(\overrightarrow{k'}) \right] &= \int \frac{d^3x d^3x' e^{ikx} e^{-ik'x'}}{\sqrt{(2\pi)^3 \, 2w_k \, (2\pi)^3 \, 2w_{k'}}} \left[\partial_0 \phi \, (x) - ik_0 \phi \, (x) \, , \; \partial_0 \phi \, (x') - ik_0' \phi \, (x') \right] \\ &= \int \frac{d^3x d^3x' e^{ikx} e^{-ik'x'}}{\sqrt{(2\pi)^3 \, 2w_k \, (2\pi)^3 \, 2w_{k'}}} \left(ik_0' \, (-i) - ik_0 i \right) \delta^3 \left(x - x' \right) \\ &= \delta^3(\overrightarrow{k} - \overrightarrow{k'}) \end{aligned}$$

Same as harmonic oscillators.

The Hamiltonian is

$$H = \int d^3k \mathcal{H}_k = \frac{1}{2} \int d^3k w_k \left[a^\dagger(\overrightarrow{k}) a(\overrightarrow{k}) + a(\overrightarrow{k}) a^\dagger(\overrightarrow{k}) \right]$$

superposition of oscillators with frequency w_k.

We can compute the commutator

$$\left[H,\ a^\dagger(k) \right] = \int d^3k'\ w_{k'}\ \left[a^\dagger(k') a(k'),\ a^\dagger(k) \right] = w_k\ a^\dagger(k)$$

If we have an eigenstate of H with eigenvalue E,

$$H|E\rangle = E|E\rangle,$$

then applying the commutator, we get

$$\left(Ha^\dagger(k) - a^\dagger(k)H \right) |E\rangle = w_k\ a^\dagger(k)|E\rangle$$

which gives

$$Ha^\dagger(k)|E\rangle = (E + w_k\ )\ a^\dagger(k)|E\rangle$$

Thus the operator $a^\dagger(k)$ will increase the energy eigenvalue by w_k, **creation operator.**

Similarly,

$$[H,\ a(k)] = \int d^3k'\ w_{k'}\ \left[a^\dagger(k')a(k'),\ a(k)\right] = -w_k\ a(k)$$

and $a(k)$ will decrease the energy eigenvalue by w_k,**destruction operator**.
From Noether's theorem, momentum operator is,

$$P_i = \int d^3x T_{0i} = \int d^3x \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \phi\right)} \partial_i \phi = \int d^3x \pi \partial_i \phi$$

and we have the commutator,

$$\begin{aligned}\left[P_i, \phi(\vec{x}, t)\right] &= \int d^3y \left[\pi(\vec{y}, t)\partial_i\phi(\vec{y}, t), \phi(\vec{x}, t)\right] \\ &= \int d^3y \partial_i \phi(\vec{y}, t) \left(-i\right) \delta^3(\vec{x} - \vec{y}) = -i\partial_i\phi(\vec{x}, t)\end{aligned}$$

In terms of creation and annihilation operators,

$$\vec{p} = \frac{1}{2} \int d^3k \vec{k} \left[a^\dagger\left(k\right) a\left(k\right) + a\left(k\right) a^\dagger\left(k\right)\right] = \int d^3k \vec{p_k}$$

with

$$\vec{p_k} = \frac{\vec{k}}{2} \left[a^\dagger\left(k\right) a\left(k\right) + a\left(k\right) a^\dagger\left(k\right)\right]$$

Note

$$a(k)\,a^{\dagger}(k) = a^{\dagger}(k)\,a(k) + \delta^{3}(0)$$

Interpret $\delta^{3}(0)$ as

$$\delta^{3}(\overrightarrow{k}) = \int \frac{d^{3}x}{(2\pi)^{3}} e^{i\overrightarrow{k}\cdot\overrightarrow{x}}$$

as $\overrightarrow{k} \to 0$

$$\delta^{3}(0) = (2\pi)^{-3} \int d^{3}x = \frac{V}{(2\pi)^{3}}$$

V total volume of the system. Then

$$H = \int d^{3}k w_{k} \left[a^{\dagger}(k)\,a(k) + \frac{(2\pi)^{-3}}{2} V \right]$$

Last term will be dropped.

To achieve this more formally, use normal ordering.

Normal ordering

In normal ordering $:(\cdots):$ move all $a^{\dagger}(k)$ to the left of $a(k)$.

For example,

.

$$
\begin{aligned}
: & \quad a(k)a^{\dagger}(k) := a^{\dagger}(k)a(k) \\
: & \quad a^{\dagger}(k)a(k) := a^{\dagger}(k)a(k)
\end{aligned}
$$

Vaccum is defined by

$$
a(k)|0\rangle = 0 \qquad \forall\ \overrightarrow{k} \qquad \Longrightarrow \langle 0|a^{\dagger}(k) = 0
$$

Then

$$
\langle 0| : f\left(a, a^{\dagger}\right) : |0\rangle = 0
$$

Define Hamiltonican by normaling ordering

$$
H = \frac{1}{2}\int d^3 k w_k : \left[a^{\dagger}(k)a(k) + a(k)a^{\dagger}(k)\right] := \int d^3 k w_k a^{\dagger}(k)a(k)
$$

Similarly,

$$
\overrightarrow{p} = \frac{1}{2}\int d^3 k \overrightarrow{p_k} : \left[a^{\dagger}(k)a(k) + a(k)a^{\dagger}(k)\right] := \int d^3 k \overrightarrow{p_k} a^{\dagger}(k)a(k)
$$

Then vacuum has zero energy and momentum.

Particle interpretation

State defined by

$$|\overrightarrow{k}\rangle = \sqrt{(2\pi)^3 2w_k} a^\dagger(k)|0\rangle$$

is eigenstate of H & $\overrightarrow{p}$,

$$H|\overrightarrow{k}\rangle = w_k|\overrightarrow{k}\rangle, \qquad \overrightarrow{p}|\overrightarrow{k}\rangle = \overrightarrow{k}|\overrightarrow{k}\rangle \qquad \text{where } w_k = \sqrt{\overrightarrow{k}^2 + \mu^2}$$

Interpret this as one-particle state because eigenvalues are related by

$$w_k^2 + \overrightarrow{k}^2 = \mu^2$$

Similarly,we can define 2 particle satate by

$$|\overrightarrow{k}_1, \overrightarrow{k}_2\rangle = \sqrt{(2\pi)^3 2w_{k_1}}\sqrt{(2\pi)^3 2w_{k_2}} a^\dagger(\overrightarrow{k_1}) a^\dagger(\overrightarrow{k_2})|0\rangle$$

Generlization to multiparticle states,

$$|\overrightarrow{k}_1, \cdots \overrightarrow{k}_n\rangle = \sqrt{(2\pi)^3 2w_{k_1}} \cdots \sqrt{(2\pi)^3 2w_{k_n}} a^\dagger(\overrightarrow{k_1}) \cdots a^\dagger(\overrightarrow{k_2})|0\rangle$$

Bose statistics

Expand arbitrary state

$$|\Phi\rangle = \left[C_0 + \sum_{i=1}^{\infty} \int d^3k_1 ... d^3k_n C_n\left(k_1, k_2, ..., k_n\right) a^{\dagger}(\overrightarrow{k_1}) ... a^{\dagger}(\overrightarrow{k_n})|0\rangle \right]$$

$C_n\left(k_1, k_2, ..., k_n\right)$ the momentum space wavefunction.
Since

$$\left[a^{\dagger}\left(k_i\right), a^{\dagger}\left(k_j\right) \right] = 0$$

$$C_n\left(k_1, ..., k_i, ..., k_j ..., k_n\right) = C_n\left(k_1, ..., k_j, ..., k_i ..., k_n\right)$$

$C_n\left(k_1, k_2, ..., k_n\right)$ satisfies Bose statistics

Scalar fields with symmetry

Suppose there are 2 scalar fields with Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu \phi_1\right)\left(\partial_\mu \phi_1\right) - \frac{\mu_1^2}{2}\phi_1^2 + \frac{1}{2}\left(\partial^\mu \phi_2\right)\left(\partial_\mu \phi_2\right) - \frac{\mu_2^2}{2}\phi_2^2$$

Here both ϕ_1, ϕ_2 are Hermitian fields. The Euler-Lagrange equations are

$$\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial^\mu \phi_i\right)}\right) - \frac{\partial \mathcal{L}}{\partial \phi_i} = 0, \qquad i = 1, 2$$

which give

$$\partial^\mu \partial_\mu \phi_1 + \mu_1^2 \phi_1 = 0, \qquad \partial^\mu \partial_\mu \phi_2 + \mu_2^2 \phi_2 = 0$$

It is clear that in the quantization, ϕ_1, ϕ_2 each will have their own creation and destruction operators, $a_1^\dagger(\overrightarrow{k_1})$ $a_2^\dagger(\overrightarrow{k_2}), a_1(\overrightarrow{k_1})$ $a_2(\overrightarrow{k_2})$.
However, when $\mu_1^2 = \mu_2^2$, the Lagrangian becomes,

$$\mathcal{L} = \frac{1}{2}\left[\left(\partial^\mu \phi_1\right)\left(\partial_\mu \phi_1\right) + \left(\partial^\mu \phi_2\right)\left(\partial_\mu \phi_2\right)\right] - \frac{\mu^2}{2}\left(\phi_1^2 + \phi_2^2\right)$$

and is invariant under the rotation in ϕ_1, ϕ_2, space,

$$\begin{aligned} \phi_1 \longrightarrow \phi_1' &= \cos\theta\ \phi_1 + \sin\theta\ \phi_2 \\ \phi_2 \longrightarrow \phi_2' &= -\sin\theta\ \phi_1 + \cos\theta\ \phi_2 \end{aligned}$$

Here θ is independent of x^μ and is usually called the global $O\left(2\right)$ symmetry. To find the conserved current, take the rotation to be infintesmal

$$\delta\phi_1 = \phi_1' - \phi_1 = \theta\ \phi_2, \qquad \delta\phi_2 = \phi_2' - \phi_2 = -\theta\ \phi_1$$

and get the conserved current,

$$j_\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu \phi_i\right)}\delta\phi_i = \left(\partial_\mu\phi_1\right)\phi_2 - \left(\partial_\mu\phi_2\right)\phi_1$$

Sometime we combine these two fields into a single complex field,

$$\phi = \frac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right)$$

and the Lagrangian becomes,

$$\mathcal{L} = \left(\partial^\mu\phi^\dagger\right)\left(\partial_\mu\phi\right) - \mu^2\phi^\dagger\phi$$

The symmetry becomes the phase transformation,

$$\phi \longrightarrow \phi' = e^{-i\theta}\phi$$

and is called the global $U\left(1\right)$ transformation. The Noether's current is then,

$$j_\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu\phi\right)}\delta\phi + \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu\phi^\dagger\right)}\delta\phi^\dagger = i[\left(\partial^\mu\phi^\dagger\right)\phi - \left(\partial^\mu\phi\right)\phi^\dagger]$$

Thus $O(2)$ symmetry is equivalent to $U\left(1\right)$ symmetry.

Fermion fields

To quantize fermion field we can proceed the same way as the scalar field. Start with Dirac equation for free particles,

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi = 0 \qquad \text{or} \qquad \overline{\psi}\left(-i\gamma^{\mu}\overleftarrow{\partial_{\mu}} - m\right) = 0$$

Lagrangian density for this equation is

$$\mathcal{L} = \bar{\psi}_{\alpha}\left(i\gamma^{\mu}\partial_{\mu} - m\right)_{\alpha\beta}\psi_{\beta}$$

Then

$$\frac{\partial\mathcal{L}}{\partial\psi^{\dagger}_{\gamma}} = \left(\gamma^{0}\right)_{\gamma\alpha}\left(i\gamma^{\mu}\partial_{\mu} - m\right)_{\alpha\beta}\psi_{\beta}, \qquad \frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\psi_{\gamma}\right)} = 0$$

and Euler-Lagrange equation gives,

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)_{\alpha\beta}\psi_{\beta} = 0$$

Conjugate momentum density is

$$\pi_{\alpha} = \frac{\partial\mathcal{L}}{\partial\left(\partial_{0}\psi_{\alpha}\right)} = i\psi^{\dagger}_{\alpha}$$

If we impose the commutation relation like scalar field, will get Dirac particles satisfying Bose statistics which is not correct physically.

Impose anticommutation relations to get Fermi-Dirac statistics,

$$\left\{\pi_\alpha\left(\overrightarrow{x},t\right),\psi_\beta\left(\overrightarrow{y},t\right)\right\}=i\delta^3\left(\overrightarrow{x}-\overrightarrow{y}\right)\delta_{\alpha\beta}\ ,\quad \left\{\psi_\alpha\left(\overrightarrow{x},t\right),\psi_\beta\left(\overrightarrow{y},t\right)\right\}=0$$

Hamiltonian density

$$\mathcal{H}=\sum_\alpha \pi_\alpha\dot{\psi}_\alpha-\mathcal{L}=i\psi^\dagger\gamma_0\gamma_0\partial_0\psi-\overline{\psi}\left(i\gamma^\mu\partial_\mu-m\right)\psi=\overline{\psi}\left(i\overrightarrow{\gamma}\cdot\overrightarrow{\nabla}+m\right)\psi$$

Mode expansion

Expansion in terms of classical solutions,

$$\psi(\vec{x},t) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left[b(p,s)\, u(p,s)\, e^{-ip\cdot x} + d^\dagger(p,s)\, v(p,s)\, e^{ip\cdot x} \right]$$

$$\psi^\dagger(\vec{x},t) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left[b^\dagger(p,s)\, u^\dagger(p,s)\, e^{ip\cdot x} + d(p,s)\, v^\dagger(p,s)\, e^{-ip\cdot x} \right]$$

Invert these relations to get the field operators in the momentum space. Multiply ψ by $u^\dagger(p',s')\, e^{ip'x}$ and integrate over x,

$$\int d^3x e^{ip'x} u^\dagger(p',s')\, \psi\left(\vec{x},t\right) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} b(p,s)\, u^\dagger(p',s')\, u(p,s)\, (2\pi)^3 \delta^3(p-p')$$

where we have used the relation,

$$u^\dagger(-p,s')\, v(p,s) = 0$$

From the Dirac equation we have

$$\bar{u}(p,s')\, \gamma^\mu (\not{p} - m)\, u(p,s) = 0$$

and

$$\bar{u}\left(p,s'\right)\left(\not{p}-m\right)\gamma^{\mu}u\left(p,s\right)=0$$

Add these two equations we get

$$p^{\mu}\bar{u}\left(p,s'\right)u\left(p,s\right)=m\bar{u}\left(p,s'\right)\gamma^{\mu}u\left(p,s\right)$$

Take the time component,

$$u^{\dagger}\left(p',s'\right)u\left(p,s\right)=2p^{0}$$

Using this relation, we get

$$b\left(p,s\right)=\int\frac{d^{3}xe^{ip\cdot x}}{\sqrt{\left(2\pi\right)^{3}2E_{p}}}u^{\dagger}\left(p,s\right)\psi\left(\vec{x},t\right)$$

The Hermitian conjugate yields

$$b^{\dagger}\left(p,s\right)=\int\frac{d^{3}xe^{-ip\cdot x}}{\sqrt{\left(2\pi\right)^{3}2E_{p}}}\psi^{\dagger}\left(\vec{x},t\right)u\left(p,s\right)$$

From these, we can compute the anti-commutation relations for b, d,

$$
\begin{aligned}
\left\{b(p,s), b^{\dagger}(p',s')\right\} &= \int \frac{d^3x' d^3x e^{ip\cdot x}}{\sqrt{(2\pi)^3 2E_p}} \frac{e^{-ip'\cdot x'}}{\sqrt{(2\pi)^3 2E_{p'}}} \left\{u^{\dagger}(p,s)\psi\left(\vec{x},t\right),\ \psi^{\dagger}(\vec{x}',t)u(p',s')\right\} \\
&= \int \frac{d^3x e^{ip\cdot x}}{\sqrt{(2\pi)^3 2E_p}} \int \frac{d^3x' e^{-ip'\cdot x'}}{\sqrt{(2\pi)^3 2E_{p'}}} (2\pi)^3 \delta^3(x-x') u^{\dagger}(p,s) u(p',s') \\
&= \delta_{ss'}\delta^3(\vec{p}-\vec{p}'),
\end{aligned}
$$

SImilarly

$$
\left\{d(p,s), d^{\dagger}(p',s')\right\} = \delta_{ss'}\delta^3(\vec{p}-\vec{p}')
$$

and all other anticommutators vanish.
Hamiltonian

$$
H = \sum_s \int d^3p \mathcal{H}_{ps}
$$

where

$$
\mathcal{H}_{ps} = E_p \left[b^{\dagger}(p,s) b(p,s) - d(p,s) d^{\dagger}(p,s)\right]
$$

Similarly,

$$
\vec{p} = \sum_s d^3p \vec{p}_p
$$

where

$$\overrightarrow{p}_p = \overrightarrow{p}\left[b^\dagger(p,s)\,b(p,s) - d(p,s)\,d^\dagger(p,s)\right]$$

Commutators of H with $b^\dagger(p,s)$

$$\left[H, b^\dagger(p,s)\right] = \sum_{s'} d^3p' \left[b^\dagger(p',s')\,b(p',s'), b^\dagger(p,s)\right] E_p = b^\dagger(p,s)\,E_p$$

$$\left[\overrightarrow{p}, b^\dagger(p,s)\right] = \overrightarrow{p}\,b^\dagger(p,s)$$

where we have used the identity

$$[AB,\ C] = A\{B,C\} - \{A,C\}B$$

$b^\dagger(p,s)$ creats a particle with E_p and $\overrightarrow{p}$ with relation $E_p = \sqrt{\overrightarrow{p}^2 + m^2}$.
$d^\dagger(p,s)$ creates a particle with same mass but opposite charge as $b^\dagger(p,s)$.

Symmetry

$$\mathcal{L} = \overline{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi$$

is invariant under,

$$\psi\left(x\right) \rightarrow e^{i\alpha}\psi\left(x\right) \Longrightarrow \psi^{\dagger}\left(x\right) \rightarrow \psi^{\dagger}\left(x\right)e^{-i\alpha} \qquad \alpha : \text{some real constant}$$

Noether's theorem,$\Longrightarrow$ conserved current,

$$\partial^{\mu}j_{\mu} = 0, \qquad \text{where} \qquad j_{\mu} = \overline{\psi}\gamma_{\mu}\psi$$

To see this consider the infinitesmal transformation

$$\delta\psi = i\alpha\psi, \qquad \delta\psi^{\dagger} = -i\alpha\psi^{\dagger}$$

Then from Noether's theorem, the conserved current is

$$j_{\mu} = \frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\psi_{\alpha}\right)}\delta\psi_{\alpha} = -i\alpha\overline{\psi}\gamma_{\mu}\psi$$

We can compute the conserved charge

$$
\begin{aligned}
Q &= \int j_0(x)\, d^3x = \int d^3x : \psi^\dagger\left(\vec{x}, t\right)\psi\left(\vec{x}, t\right) : \\
&= \int d^3x \sum_{ss'} \int \frac{d^3p'}{\sqrt{(2\pi)^3 2E_{p'}}} : \left[b^\dagger\left(p', s'\right) u^\dagger\left(p', s'\right) e^{ip'\cdot x} + d\left(p', s'\right) v^\dagger\left(p', s'\right) e^{-ip'\cdot x}\right] \\
&\quad \times \int \frac{d^3p}{\sqrt{(2\pi)^3 2E_p}} \left[b(p,s)\, u(p,s)\, e^{-ip\cdot x} + d^\dagger(p,s)\, v(p,s)\, e^{ip\cdot x}\right] : \\
&= \sum_s \int d^3p : \left[b^\dagger(p,s)\, b(p,s) + d(p,s)\, d^\dagger(p,s)\right] := \sum \int d^3p \left[N^+(p,s) - N^-(p,s)\right]
\end{aligned}
$$

where

$$
N^+_{ps} = b^\dagger(p,s)\, b(p,s) \qquad N^-_{ps} = d^\dagger(p,s)\, d(p,s)
$$

are the number operators $\Longrightarrow$ particle and anti-particle have opposite "charge".

Electromagnetic fields

Maxwell's equations,

$$\nabla \cdot \overrightarrow{B} = 0, \quad \nabla \times \overrightarrow{E} + \frac{\partial \overrightarrow{B}}{\partial t} = 0, \tag{2}$$

$$\nabla \cdot \vec{E} = 0, \qquad \frac{1}{\mu_0} \nabla \times \overrightarrow{B} - \epsilon_0 \frac{\partial \overrightarrow{E}}{\partial t} = 0 \tag{3}$$

Introduce $\overrightarrow{A}, \phi$ by

$$\overrightarrow{B} = \nabla \times \overrightarrow{A}, \qquad \overrightarrow{E} = -\nabla \phi - \frac{\partial \overrightarrow{A}}{\partial t} \tag{4}$$

These solve equations in Eq(2). Write relations in Eq(4) as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad \text{with} \quad F^{0i} = \partial^0 A^i - \partial^i A^0 = -E^i, \quad F^{ij} = \partial^i A^j - \partial^j A^i = -\epsilon_{ijk} B_k$$

Other two sets of equations in Eq(3)

$$\partial_\nu F^{\mu\nu} = 0, \quad \mu = 0, 1, 2, 3$$

For example

$$\mu = 0, \quad \partial_i F^{0i} = 0 \quad \Rightarrow \quad \nabla \cdot \overrightarrow{E} = 0$$

$$\mu = i, \quad \partial_\nu F^{i\nu} = 0 \quad \Rightarrow \quad \nabla \times \overrightarrow{B} - \frac{\partial \overrightarrow{E}}{\partial t} = 0$$

Note $c^2 = \dfrac{1}{\mu_0 \epsilon_0} = 1$. $F^{\mu\nu}$ is invariant under the transformation,

$$A^\mu \longrightarrow A^\mu + \partial^\mu \alpha \qquad \alpha = \alpha(x)$$

$\alpha(x)$ is arbitrary function. This is called <u>gauge transformation</u>. Given a set of $\overrightarrow{B}$ and $\overrightarrow{E}$ fields, $\overrightarrow{A}$,and ϕ are not unique. Different $\alpha(x)$ gives same $\overrightarrow{B}$ and $\overrightarrow{E}$ fields This property is usually called the **gauge invariance**.

Lagrangian density given by,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (\overrightarrow{E}^2 - \overrightarrow{B}^2)$$

will give Maxwell equations a la Euler-Lagrange equations.. To see this we compute

$$\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A_\nu \right)} = - \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right), \qquad \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

Then

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A_\nu \right)} = \frac{\partial \mathcal{L}}{\partial A_\nu}, \qquad \Longrightarrow \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right) = \partial_\mu F^{\mu\nu} = 0$$

These are indeed the Maxwell equations as we discussed before.
Conjugate momenta

$$\pi_0 = \frac{\partial L}{\partial(\partial_0 A_0)} = 0, \qquad \pi^i(x) = \frac{\partial L}{\partial(\partial_0 A_i)} = -F^{0i} = E^i$$

No conjugate momenta for $A_0 \Longrightarrow$ not a dynamical degree of freedom.
Hamiltanian density,

$$\mathcal{H} = \pi^k \dot{A}_k - \mathcal{L} = \frac{1}{2} (\overrightarrow{E}^2 + \overrightarrow{B}^2) + (\overrightarrow{E} \cdot \nabla) A_0$$

Using $\overrightarrow{\nabla} \cdot \overrightarrow{E} = 0$,Hamiltonian becomes,

$$H = \int d^3x\mathcal{H} = \frac{1}{2}\int d^3x(\overrightarrow{E}^2 + \overrightarrow{B}^2)$$

Impose commutation relation,

$$[\pi^i(\overrightarrow{x},t),\ A^j(\overrightarrow{y},t)] = -i\delta_{ij}\delta^3(\overrightarrow{x} - \overrightarrow{y}), \quad ...$$

But this is not consistent with $\overrightarrow{\nabla} \cdot \vec{E} = 0$ because

$$[\nabla \cdot E(x,t),\ A_j(x,t)] = -i\partial_j\delta^3(x-y) \neq 0$$

$\delta-$function in momentum space

$$\partial_j\delta^3(\overrightarrow{x} - \overrightarrow{y}) = i\int \frac{d^3k}{(2\pi)^3}e^{i\overrightarrow{k}\cdot(\overrightarrow{x}-\overrightarrow{y})}k_j$$

To get zero for the commutator of $\nabla \cdot E$, replace,

$$\delta_{ij}\delta^3(\vec{x}-\vec{y}) \longrightarrow \delta^{tr}_{ij}(\vec{x}-\vec{y}) = \int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}(\delta_{ij} - \frac{k_ik_j}{k^2})$$

then

$$\partial_i\delta^{tr}_{ij}\delta^3(\vec{x}-\vec{y}) = i\int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}(\cdot\vec{x}-\vec{y})}k_i(\delta_{ij} - \frac{k_ik_j}{k^2}) = 0$$

So commutator is modified to,

$$[E^i(x,t),\ A_j(y,t)] = -i\delta^{tr}_{ij}(\vec{x}-\vec{y})$$

which implies

$$[E^i(x,t),\ \overrightarrow{\nabla}\cdot\vec{A}(y,t)] = 0$$

Now that A_0 and $\overrightarrow{\nabla}\cdot\vec{A}$ commute with all operators, they must be C-number. Choose a gauge such that

$$A_0 = 0 \textit{ and } \nabla\cdot\vec{A} = 0 \qquad \text{radiation gauge}$$

In this gauge

$$\pi^i = \partial^i A^0 - \partial^0 A^i = -\partial^0 A^i$$

$$[\partial_0 A^i(\vec{x},t),\ A^j(\vec{y},t)] = i\delta^{tr}_{ij}(\vec{x}-\vec{y})$$

Mode expansion

Equation of motion $\partial_\nu F^{\mu\nu} = 0$ gives

$$\partial_\nu \left(\partial^\nu A^\mu - \partial^\mu A^\nu \right) = \Box A^\mu - \partial^\mu \left(\partial_\upsilon A^\upsilon \right) = 0$$

In radiaiton gauge ,

$$A_0 = 0, \qquad \overrightarrow{\nabla} \cdot \overrightarrow{A} = 0$$

wave equation becomes

$$\Box \, \overrightarrow{A} = 0 \qquad \text{massless Klein-Gordon equation}$$

General solution

$$\overrightarrow{A}(\vec{x}, t) = \int \frac{d^3 k}{\sqrt{2\omega(2\pi)^3}} \sum_\lambda \overrightarrow{\epsilon}(\vec{k}, \lambda)[a(k, \lambda)e^{-ikx} + a^\dagger(k, \lambda)e^{ikx}] \qquad w = k_0 = |\overrightarrow{k}|$$

Only two degrees of freedom

$$\vec{\epsilon}(k, \lambda), \lambda = 1, 2 \quad \text{with } \vec{k} \cdot \vec{\epsilon}(k, \lambda) = 0$$

Standard choice

$$\vec{\epsilon}(k, \lambda) \cdot \vec{\epsilon}(k, \lambda') = \delta_{\lambda\lambda'}, \quad \vec{\epsilon}(-k, 1) = -\vec{\epsilon}(k, 1), \quad \vec{\epsilon}(-k, 2) = \vec{\epsilon}(-k, 2)$$

Solve for $a(k,\lambda)$ and $a^+(k,\lambda)$

$$a(k,\lambda) = i\int \frac{d^3x}{\sqrt{(2\pi)^3 2\omega}}[e^{ik\cdot x}\overleftrightarrow{\partial_0}\vec{\epsilon}(k,\lambda)\cdot\vec{A}(x)]$$

$$a^\dagger(k,\lambda) = -i\int \frac{d^3x}{\sqrt{(2\pi)^3 2\omega}}[e^{-ik\cdot x}\overleftrightarrow{\partial_0}\vec{\epsilon}(k,\lambda)\cdot\vec{A}(x)]$$

Commutation relations,

$$[a(k,\lambda),\ a^\dagger(k',\lambda')] = \delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}'), \quad [a(k,\lambda),\ a(k',\lambda')] = 0,$$

Hamiltonian and momentum operators are

$$H = \frac{1}{2}\int d^3x : (E^2+B^2) := \int d^3k\omega\sum_\lambda a^+(k,\lambda)a(k,\lambda)$$

$$\vec{P} = \int d^3x : E\times B := \int d^3k\vec{k}\sum_\lambda a^+(k,\lambda)a(k,\lambda)$$

The vaccum is defined by

$$a(\vec{k},\lambda)|0>= 0 \quad \forall\vec{k},\lambda$$

and one photon state with momentum k polarization λ is given by, $a^\dagger(\vec{k},\lambda)|0>$.

Appendix 1– Simple Harmonic Oscillator

Here we review the creation and annihilation operators in the simple harmonic oscillator in one dimension. The Hamiltonian is

$$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 x^2$$

where for convenience we have set $m = 1$. Here p, x satisfy the comutation relation,

$$[x, p] = i$$

Define

$$a = \sqrt{\frac{1}{2\omega}}\,(\omega x + ip)\,, \qquad a^{\dagger} = \sqrt{\frac{1}{2\omega}}\,(\omega x - ip)$$

The commutator is ,

$$\left[a, a^{\dagger}\right] = \frac{1}{2\omega}\,[\omega x + ip,\ \omega x - ip] = 1$$

From

$$x = \frac{1}{\sqrt{2\omega}}\left(a + a^{\dagger}\right), \qquad p = -i\sqrt{\frac{\omega}{2}}\left(a - a^{\dagger}\right)$$

we get for the Hamiltonian

$$H = \frac{1}{2}[-\frac{\omega}{2}\left(a - a^{\dagger}\right)^2 + \frac{\omega^2}{2\omega}\left(a + a^{\dagger}\right)^2 = \frac{\omega}{2}\left(a^{\dagger}a + aa^{\dagger}\right)$$

Using the commutation relation we can write H as

$$H = \omega \left(a^\dagger a + \frac{1}{2} \right)$$

The second term here is called the zero-point energy.

We can compute the commutator of H with a or $a^\dagger$,

$$[H,\ a] = -\omega a, \qquad \left[H,\ a^\dagger\right] = \omega a^\dagger$$

Suppose $|E\rangle$ is eigenstate of Hamiltoian with eigenvalue E,

$$H\,|E\rangle = E\,|E\rangle$$

Then we get

$$\left(Ha^\dagger - a^\dagger H\right)|E\rangle = \omega a^\dagger\,|E\rangle\,, \qquad \Longrightarrow \qquad H\left(a^\dagger\,|E\rangle\right) = (E+\omega)\left(a^\dagger\,|E\rangle\right)$$

Thus $a^\dagger$ increases the energy eigenvalue by ω and is called **raising operator** (or **creation operator**). Similarly,

$$(Ha - aH)\,|E\rangle = -\omega a\,|E\rangle\,, \qquad \Longrightarrow \qquad H\,(a\,|E\rangle) = (E-\omega)\,(a\,|E\rangle)$$

which implies that the operator a decreaes the energy eigenvalue by ω.Since H is bounded below, there must exist a state with lowest energy eigen value, the ground state $|0\rangle$, defined by

$$a\,|0\rangle = 0$$

will have energy eigen value

$$H|0\rangle = \frac{1}{2}\omega|0\rangle$$

It is clear that the excited states are related to $|0\rangle$ by the action of $a^\dagger$. For example,

$$H|n\rangle = \left(n + \frac{1}{2}\right)\omega|n\rangle, \qquad \text{where} \qquad |n\rangle = \frac{\left(a^\dagger\right)^n}{\sqrt{n!}}|0\rangle$$

The state $|n\rangle$ can be interpreted as state with n quanta, each with energy ω. So the operator $N = a^\dagger a$ is the number operator.

Appendix 2–U(1) local symmetry

The free Maxwll's equations are

$$\vec{\nabla} \cdot \vec{B} = 0, \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0,$$

$$\vec{B} = \nabla \times \vec{A}, \qquad \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

Solve the first two equations by introducing $\vec{A}, \phi$

$$\vec{B} = \nabla \times \vec{A}, \qquad \vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} \tag{5}$$

Convenient to write

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad \text{with} \quad F^{0i} = \partial^0 A^i - \partial^i A^0 = -E^i, \quad F^{ij} = \partial^i A^j - \partial^j A^i = -\epsilon_{ijk} B_k$$

For a charged particle moving in electromagnetic field, the equation of motion is,

$$m\frac{d^2\vec{x}}{dt^2} = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

The Lagrangian for these equations is

$$L = \frac{1}{2} m \left(\vec{v} \right)^2 + e \vec{A} \cdot \vec{v} - e A_0$$

To see this, we compute the derivatives with respect to $\vec{x}$ and $\vec{v}$,

$$\frac{\partial L}{\partial v_i} = m v_i + e A_i, \quad \frac{\partial L}{\partial x_i} = e \frac{\partial A_j}{\partial x_i} v_j - e \frac{\partial A_0}{\partial x_i}$$

and

$$\frac{d}{dt} \left(\frac{\partial L}{\partial v_i} \right) = m \frac{d v_i}{dt} + e \frac{\partial A_i}{\partial x_j} \frac{d x_j}{dt} + e \frac{\partial A_i}{\partial t}$$

Euler-Lagrange equation gives

$$m \frac{d v_i}{dt} + e \frac{\partial A_i}{\partial x_j} \frac{d x_j}{dt} + e \frac{\partial A_i}{\partial t} = e \frac{\partial A_j}{\partial x_i} v_j - e \frac{\partial A_0}{\partial x_i}$$

On the other hand,

$$\left(\vec{v} \times \vec{B} \right)_i = \varepsilon_{ijk} v_j B_k = \varepsilon_{ijk} v_j \varepsilon_{klm} \partial_l A_m = v_j \left(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl} \right) \partial_l A_m = v_j \left(\partial_i A_j - \partial_j A_i \right)$$

Then we get

$$m \frac{d v_i}{dt} = -e \frac{\partial A_i}{\partial x_j} v_j - e \frac{\partial A_i}{\partial t} + e \frac{\partial A_j}{\partial x_i} v_j - e \frac{\partial A_0}{\partial x_i}$$

Or

$$m\frac{dv_i}{dt} = e\left(\partial_i A_j - \partial_j A_i\right) v_j + e\left(-\partial_i A_0 - \partial_0 A_i\right) = e\left(\vec{E} + \vec{v} \times \vec{B}\right)_i$$

which is the correct equation of motion.
From Lagrangian define the conjugate momentum,

$$p_i = \frac{\partial L}{\partial v_i} = mv_i + eA_i, \qquad \Longrightarrow \qquad v_i = \frac{1}{m}\left(p_i - eA_i\right)$$

The Hamiltonian is then

$$\begin{aligned} H &= p_i v_i - L = p_i v_i - \frac{1}{2}m\left(\vec{v}\right)^2 - e\vec{A}\cdot\vec{v} + eA_0 \\ &= \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 + eA_0 \end{aligned}$$

Note that we can obain this Hamitonian fprm the free Hamiltonian $H = \vec{p}^{\,2}/2m$ by the substition,

$$\vec{p} \longrightarrow \vec{p} - e\vec{A}, \qquad H \longrightarrow H - eA_0$$

Or

$$p^\mu \longrightarrow p^\mu - eA^\mu$$

This is usally called the **principle of minimal substitution.**

The Schrodinger equation for a charged particle moving in the electromagnetic field is,

$$[-\frac{1}{2m}\left(\overrightarrow{\nabla} - ie\overrightarrow{A}\right)^2 + eA_0]\psi = i\frac{\partial\psi}{\partial t}$$

This shows that it is the potentials $\overrightarrow{A}, A_0$,not the $\overrightarrow{E}$, $\overrightarrow{B}$ fields show up in the Schrodinger equation. However, Schrodinger equation is not invariant under the gauge transformation,

$$A^\mu \longrightarrow A^\mu + \partial^\mu\alpha, \qquad \text{or} \qquad \overrightarrow{A} \longrightarrow \overrightarrow{A} - \overrightarrow{\nabla}\alpha, \qquad A_0 \longrightarrow A_0 + \partial_0\alpha$$

But it turns out that we can recover the Schrodinger equation if we also change the wave function ψ by a phase,

$$\psi \longrightarrow \psi' = e^{-ie\alpha}\psi$$

This can be seen as follows. Define the covariant derivative as

$$\overrightarrow{D}\psi = \left(\overrightarrow{\partial} - ie\overrightarrow{A}\right)\psi$$

The covariant derivative for the new fields is then,

$$\begin{aligned}\overrightarrow{D}\psi' &= \left(\overrightarrow{\partial} - ie\overrightarrow{A}'\right)\psi' = e^{-ie\alpha}[\overrightarrow{\partial} - ie\overrightarrow{\nabla}\alpha - ie\left(\overrightarrow{A} - \overrightarrow{\nabla}\alpha\right)]\psi \\ &= e^{-ie\alpha}\left(\overrightarrow{D}\psi\right)\end{aligned}$$

So the covariant derivative $\vec{D}\psi$ transforms by a phase in the same way as the field ψ.In other words, the covariant derivative $\vec{D} = \left(\vec{\partial} - ie\vec{A}\right)$ does not change the transformation property of the object it acts on. It is then easy to see that

$$\vec{D}^2 \psi' = e^{-ie\alpha} \left(\vec{D}^2 \psi\right)$$

For the time derivative, we have

$$D_0\psi = (\partial_0 + ieA_0)\,\psi$$

and

$$D_0\psi' = e^{-ie\alpha}\,(\partial_0 + ie\partial_0\alpha - ieA_0 - ie\partial_0\alpha)\,\psi = e^{-ie\alpha} D_0\psi$$

With this phase transformation, the Schrodinger equation

$$[-\frac{1}{2m}\left(\vec{\nabla} - ie\vec{A}'\right)^2 + eA_0']\psi' = i\frac{\partial\psi'}{\partial t}$$

becomes

$$e^{-ie\alpha}[-\frac{1}{2m}\left(\vec{\nabla} - ie\vec{A}\right)^2 + eA_0]\psi = e^{-ie\alpha} i\frac{\partial\psi}{\partial t}$$

After cancelling out the phase $e^{-ie\alpha}$, we get back the original Schrodinger equation.
The phase transformation of the wave function is a symmetry transformation and is a local symmetry because $\alpha = \alpha\left(\vec{x}, t\right)$. The phase transformation in usually referred to as $U(1)$ transformation and we call the elecromagnetic possesses $U(1)$ local symmetry.