Quantum Field Theory
Homework set 2, Due Wed Oct 29

1. The Dirac Hamiltonian for free particle is given by

$$H = \vec{\alpha} \cdot \vec{p} + \beta m$$

The angular momentum operator is of the form,

$$\vec{L} = \vec{r} \times \vec{p}$$

   (a) Compute the commutators,

$$\left[\vec{L}, H\right]$$

   Is $\vec{L}$ conserved?

   (b) Define $\vec{S} = -\frac{i}{4}\left(\vec{\alpha} \times \vec{\alpha}\right)$ and show that

$$\left[\vec{L} + \vec{S},\ H\right] = 0$$

   (c) Show that $\vec{S}$ satisfy the angular momentum algebra, i.e.

$$[S_i, S_j] = i\varepsilon_{ijk} S_k$$

   and

$$\vec{S}^2 = \frac{3}{4}.$$

2. The Dirac spinors are of the form,

$$u(p,s) = \sqrt{E+m}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_s, \qquad v(p,s) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\chi_s \qquad s = 1,2$$

where

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   (a) Show that

$$\bar{u}(p,s)\,u(p,s') = 2m\delta_{ss'}, \qquad \bar{v}(p,s)\,v(p,s') = 2m\delta_{ss'}$$
$$\bar{v}(p,s)\,u(p,s') = 0, \qquad \bar{u}(p,s)\,v(p,s') = 0$$
$$v^\dagger(-p,s)\,u(p,s') = 0, \qquad u^\dagger(p,s)\,v(-p,s') = 0$$

   (b) Show that

$$\sum_s u_\alpha(p,s)\,\bar{u}_\beta(p,s) = (\not{p} + m)_{\alpha\beta}$$
$$\sum_s v_\alpha(p,s)\,\bar{v}_\beta(p,s) = (\not{p} - m)_{\alpha\beta}$$

3. Suppose a free Dirac particle at t=0, is described by a wavefunction,

$$\psi\left(0, \vec{x}\right) = \frac{1}{(\pi d^2)^{3/4}} \exp\left(-\frac{r^2}{2d^2}\right)\omega$$

where d is some constant and

$$\omega = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Compute the wavefunction for $t \neq 0$. What happens when d is very small.

4. Consider a 2×2 hermitian matrix defined by

$$X = x_0 + \vec{\sigma} \cdot \vec{x}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices and $\left(x_0, \vec{x}\right)$ are space-time coordinates.

(a) Compute the determinant of X

(b) Suppose U is a 2×2 matrix with $\det U = 1$. Define a new 2×2 matrix by a similarity transformation,

$$X' = UXU^\dagger$$

Show that X' can be written as

$$X' = x_0' + \vec{\sigma} \cdot \vec{x}'$$

(c) Show that the relation between $\left(x_0, \vec{x}\right)$ and $\left(x_0', \vec{x}'\right)$ is a Lorentz transformation.

(d) Suppose U is of the form,

$$U = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}$$

Find the relation between $\left(x_0, \vec{x}\right)$ and $\left(x_0', \vec{x}'\right)$.

5. Dirac particle in the presence of electromagnetic field satisfies the equation,

$$\left[\gamma^\mu \left(i\partial_\mu - eA_\mu\right) - m\right] \psi(x) = 0$$

Or

$$i\frac{\partial \psi}{\partial t} = \left[\vec{\alpha} \cdot \left(\vec{p} - e\vec{A}\right) + \beta m + e\phi\right] \psi$$

In the non-relativistic limit, we can write

$$\psi(x) = e^{-imt} \begin{pmatrix} u \\ l \end{pmatrix}$$

Show that the upper component satisfies the equation,

$$i\frac{\partial u}{\partial t} = \left[\frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^2 - \frac{e}{m}\vec{\sigma} \cdot \vec{B} + e\phi\right] u$$

For the case of weak uniform magnetic field $\vec{B}$ we can take $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$. Show that

$$i\frac{\partial u}{\partial t} = \left[\frac{1}{2m}\left(\vec{p}\right)^2 - \frac{e}{2m}\left(\vec{L} + 2\vec{S}\right) \cdot \vec{B}\right] u.$$

6. $a_1^\dagger, a_2^\dagger, a_1, a_2$ are creation and annihilation operators satifying the commutation relations

$$\left[a_i, a_j^\dagger\right] = \delta_{ij}, \qquad [a_i, a_j] = 0, \qquad i, j = 1, 2$$

Define

$$J_+ = a_1^\dagger a_2, \qquad J_- = (J_+)^\dagger, \qquad J_3 = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right)$$

(a) Compute the commutators

$$[J_x, J_y], \qquad [J_y, J_z], \qquad [J_z, J_x]$$

where $J_x \equiv \frac{1}{2}(J_+ + J_-)$, $\quad J_y \equiv \frac{1}{2i}(J_+ - J_-)$

(b) Define the state $|0\rangle$ by

$$a_i |0\rangle = 0, \qquad \text{for } i = 1, 2$$

Let the state $|n_1, n_2\rangle$ be

$$|n_1, n_2\rangle = \frac{1}{\sqrt{n_1! n_2!}} \left(a_1^\dagger\right)^{n_1} \left(a_2^\dagger\right)^{n_2} |0\rangle$$

Show that this state is an eigenstate of J_3 and compute the eigenvalue.

(c) Show that this is also eigen state of $J^2 = J_1^2 + J_2^2 + J_3^2$ and compute the eigenvalue.

(d) Show that the state $J_+ |n_1, n_2\rangle$ is an eigenstate of J_3. What is the eigenvalue?