Quantum Field Theory
Homework set 1, due Wed Oct 15

1. Show that the combination
$$\frac{d^3p}{2E}, \qquad \text{with } \; E = \sqrt{\vec{p}^{\,2} + m^2}$$
which occurs frequently in phase space calculation integration is invariant under Lorentz transformation.

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2. Consider a system where 2 particles interacting with eac other through potential energy $V\left(\vec{x}_1 - \vec{x}_2\right)$ so that the Lagrangian is of the form,
$$L = \frac{m_1}{2}\left(\frac{d\vec{x}_1}{dt}\right)^2 + \frac{m_2}{2}\left(\frac{d\vec{x}_2}{dt}\right)^2 - V\left(\vec{x}_1 - \vec{x}_2\right)$$
   (a) Show that this Lagrangian is invariant under the spatial translation given by
$$\vec{x}_1 \to \vec{x}\,'_1 = \vec{x}_1 + \vec{a}, \qquad \vec{x}_2 \to \vec{x}\,'_2 = \vec{x}_2 + \vec{a},$$
   where $\vec{a}$ is an arbitrary vector.

   (b) Use Noether's theorem to construct the conserved quantity corresponding to this symmetry.

3. Compute the following physical quantities in the right units.

   (a) The total cross section for $e^+e^- \to \mu^+\mu^-$ at high energies is of the form,
$$\sigma\left(e^+e^- \to \mu^+\mu^-\right) = \frac{4\pi\alpha^2}{3s}, \qquad s = 4E^2, \qquad E \text{ :energy of } e^- \text{ in cm frame, } \; \alpha \text{ fine structure constant}$$
   Compute the cross section for the energies $E = 100Gev$, $7Tev$

   (b) The formula for the μ decay is given by
$$\Gamma\left(\mu \to e\nu\bar{\nu}\right) = \frac{G_F^2 M_P^3}{192\pi^3}, \qquad G_F \; \text{ is the Fermi constant,} \qquad M_P \text{ the proton mass}$$
   Compute the muon lifetime in seconds.

4. Construct the Lorentz transformation for motion of coordinate axis in arbitrary direction by using the fact that coordinates perpendicualr to the direction of motion remain unchanged.

5. Electric and magnetic fields, $\vec{E}$, $\vec{B}$, combine into an antisymmetric second rank tensor unde the Lorentz transformtion,
$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad \text{with} \quad F^{0i} = \partial^0 A^i - \partial^i A^0 = -E^i, \quad F^{ij} = \partial^i A^j - \partial^j A^i = -\epsilon_{ijk} B_k$$
   These Minkowski tensors have the following property under the Lorentz transformation,
$$F^{\mu\nu} \to F'^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta}, \qquad \Lambda^\mu_\alpha : \text{ matrix element of Lorentz transformation}$$
   Suppose an inertial frame O' moves with respect to O with velocity v in the positive x-direction.

   (a) Find the relations between the electric and magnetic fields, $\vec{E}'$, $\vec{B}'$,in the O' and those in the O frame.

   (b) Show that the combination $\vec{E} \cdot \vec{B}$, does not change from O to O' frames.

   (c) Show that the combination $\vec{E}^{\,2} - \vec{B}^{\,2}$, does not change either.