Quantum Field Theory

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Path integral formalism has close relationship to classical dynamics, e.g. the transition amplitude

$$\langle f|i\rangle = \int [dx]\, e^{iS/\hbar}$$

as $\hbar \to 0$, the trajectory with smallest S dominates, the action principle. Here uses the ordinary functions not the operators. Later in non-Abelian gauge theory, to remove unphysical degrees of freedom can be accomodated in the path integral formalism by imposing constraints in the integral.

Quantum Mechanics in 1-dimension

In QM, transition from $|q, t\rangle$ to $\langle q', t'|$, can be written as,

$$\langle q' t'|qt\rangle = \langle q'|e^{-iH(t-t')}|q\rangle$$

where $|q\rangle's$ are eigenstates of position operator Q in the Schrodinger picture,

$$Q|q\rangle = q|q\rangle$$

and $|q, t\rangle$ denotes corresponding state in Heisenberg picture,

$$|q, t\rangle = e^{iHt}|q\rangle$$

Ipath integral formalism,this can be written as

$$\langle q' t'|qt\rangle = N \int [dq] exp\{i \int_t^{t'} d\tau L(q, \dot{q})\}$$

To get this formula, divide the interval $(t\prime, t)$ into n intervals ,

$$\delta t = \frac{t' - t}{n}$$

and write ,

$$\langle q'|e^{-iH(t'-t)}|q\rangle = \int dq_1 ... dq_{n-1} \langle q'|e^{-iH\delta t}|q_{n-1}\rangle\langle q_{n-1}|e^{-iH\delta t}|q_{n-2}\rangle ... \langle q_1|e^{-iH\delta t}|q\rangle$$

If we take the Hamiltonian to be in the simple form,

$$H(P, Q) = \frac{p^2}{2m} + V(Q)$$

then

$$\begin{aligned}
\langle q_j|H|q_i\rangle &= \langle q_j|\frac{p^2}{2m}|q_i\rangle + V(\frac{q_i + q_j}{2})\delta(q_i - q_j) \\
&= \int \langle q_j|\frac{p^2}{2m}|p_k\rangle\langle p_k|q_i\rangle(\frac{dp_k}{2\pi}) + V(\frac{q_i + q_j}{2}) \int \frac{dp_k}{2\pi} e^{ip_k(q_j - q_i)} \\
&= \int \frac{dp_k}{2\pi} e^{ip_k(q_j - q_i)}[\frac{p_k^2}{2m} + V(\frac{q_i + q_j}{2})]
\end{aligned}$$

where we have used

$$\langle p|q\rangle = e^{-ipq}$$

which is the momentum eigenfunction in coordinate space. Exponentiation of this infinitesmal result gives

$$\langle q_j|e^{-iH\delta t}|q_i\rangle \simeq \int \frac{dp_k}{2\pi} e^{ip_k(q_j-q_i)}\{1-i\delta t[\frac{p_k^2}{2m}+V(\frac{q_i+q_j}{2})]\} \simeq \int \frac{dp_k}{2\pi} e^{ip_k(q_j-q_i)} \exp\{-i\delta t[\frac{p_k^2}{2m}+V(\frac{q_i+q_j}{2})]\}$$

The whole transition matrix element can then be written as

$$\langle q'|e^{-iH(t'-t)}|q\rangle \cong \int (\frac{dp_1}{2\pi})...(\frac{dp_n}{2\pi}) \int dq_1...dq_{n-1} \exp\{i\left[\sum_{i=1}^{n} p_i(q_i-q_{i-1})-(\delta t)H(p_i,\frac{q_i+q_{i+1}}{2})\right]\}$$

This can be written formally as

$$\langle q'|e^{-iH(t'-t)}|q\rangle = \int [\frac{dpdq}{2\pi}] exp\{i\int_t^{t'} dt[p\dot{q}-H(p,q)]\}$$

$$\equiv \lim_{n\to\infty} \int (\frac{dp_1}{2\pi})...(\frac{dp_n}{2\pi}) \int dq_1...dq_{n-1} exp\{i\sum_{i=1}\delta t[p_i(\frac{q_i-q_{i-1}}{\delta t})-H(p_i,\frac{q_i+q_{i+1}}{2})]\}$$

If Hamiltonian depends quadractically on p, use the formula

$$\int_{-\infty}^{+\infty} \frac{dx}{2\pi} e^{-ax^2+bx} = \frac{1}{\sqrt{4\pi a}} e^{\frac{b^2}{4a}}$$

to get

$$\int \frac{dp_i}{2\pi} \exp[\frac{-i\delta t}{2m}p_i^2+ip_i(q_i-q_{i-1})] = (\frac{m}{2\pi i\delta t})^{1/2} exp[\frac{im(q_i-q_{i-1})^2}{2\delta t}]$$

Then

$$\langle q'|e^{-iH(t'-t)}|q\rangle = \lim_{n\to\infty}(\frac{m}{2\pi i\delta t})^{n/2}\int\prod_{i=1}^{n-1}dq_i\exp\{i\sum_{i=1}^{n}\delta t[\frac{m}{2}(\frac{q_i-q_{i-1}}{\delta t})^2-V]\}$$

or

$$\langle q't'|qt\rangle = \langle q'|e^{-iH(t'-t)}|q\rangle = N\int[dq]\exp\{i\int_t^{t'}d\tau[\frac{m}{2}\dot{q}^2-V(q)]\}$$

This is the path integral representation for amplitude from initial state $|q,t\rangle$ to final state $\langle q',t'|$. Or

$$\langle q't'|qt\rangle = N\int[dq]\exp iS$$

Green's functions

To generalize this to field theory where we have vacuum expectation value of field operators, we consider

$$G(t_1, t_2) = \langle 0|T(Q^H(t_1)Q^H(t_2))|0\rangle$$

Inserting complete sets of states,

$$G(t_1, t_2) = \int dqdq' \langle 0|q', t'\rangle\langle q', t'|T(Q^H(t_1)Q^H(t_2))|q, t\rangle\langle q, t|0\rangle$$

The matrix element

$$\langle 0|q, t\rangle = \phi_0(q)e^{-iE_0 t} = \phi_0(q, t)$$

is the wavefunction for ground state. Consider the case

$$t' > t_1 > t_2 > t,$$

we can write

$$\langle q', t'|T(Q^H(t_1)Q^H(t_2))|q, t\rangle = \langle q'|e^{-iH(t'-t_1)}Q^s e^{-iH(t_1-t_2)}Q^s e^{-iH(t_2-t)}|q\rangle$$

$$= \int \langle q'|e^{-iH(t'-t_1)}|q_1\rangle q_1\langle q_1|e^{-iH(t_1-t_2)}|q_2\rangle q_2\langle q_2|e^{-iH(t_2-t)}|q\rangle dq_1 dq_2$$

$$= \int [\frac{dpdq}{2\pi}]q_1(t_1)q_2(t_2)exp\{i\int_t^{t'} d\tau[p\dot{q} - H(p, q)]\}$$

For the other time sequence

$$t' > t_2 > t_1 > t,$$

we get same formula, because path integral orders the time sequence automatically through the division of time interval into small pieces. The Green's function is then

$$G(t_1,t_2) = \int dqdq'\phi_0(q',t')\phi_0^*(q,t)\int[\frac{dpdq}{2\pi}]q_1(t_1)q_2(t_2)exp\{i\int_t^{t'}d\tau[p\dot{q}-H(p,q)]\} \tag{1}$$

We remove wavefunction $\phi_0(q,t)$ by the following procedure. Write

$$\langle q',t'|\theta(t_1,t_2)|q,t\rangle = \int dQdQ'\langle q',t'|Q',T'\rangle\langle Q',T'|\theta(t_1,t_2)|Q,T\rangle\langle Q,t|q,t\rangle$$

where

$$\theta(t_1,t_2) = T(Q^H(t_1)Q^H(t_2))$$

Let $|n>$ be eigenstate with energy E_n and wave function ϕ_n, i.e.,

$$H|n>= E_n|n>, \quad \langle q|n\rangle = \phi_n^*(q)$$

Then

$$\langle q',t'|Q',t'\rangle = \langle q'|e^{-iH(t'-T')}|Q'\rangle = \sum_n\langle q'|n\rangle e^{-iE_n(t'-T')}\langle n|Q'\rangle = \sum_n\phi_n^*(q')\phi_n(Q')e^{-iE_n(t'-T')}$$

To isolate the ground state wavefunction, take an "unusual limit",

$$\lim_{t'\to -i\infty}\langle q',t'|Q',T'\rangle = \phi_0^*(q')\phi_0(Q')e^{-E_0|t'|}e^{iE_0T'}$$

Similarly,

$$\lim_{t\to i\infty}\langle Q,T|q,t\rangle = \phi_0(q)\phi_0^*(Q)e^{-E_0|t|}e^{-iE_0T}$$

With these we write

$$\lim_{\substack{t'\to -i\infty \\ t\to i\infty}}\langle q',t'|\theta(t_1,t_2)|q,t\rangle = \int dQdQ'\phi_0^*(q')\phi_0(Q')\langle Q',T'|\theta(t_1,t_2)|Q,T\rangle\phi_0^*(Q)\phi_0(q)e^{-E_0|t'|}e^{iE_0T'}e^{-iE_0T}e^{-E_0|t|}$$

$$= \phi_0^*(q')\phi_0(q)e^{-E_0|t'|}e^{-E_0|t|}G(t_1,t_2)$$

It is easy to see that

$$\lim_{\substack{t'\to -i\infty \\ t\to i\infty}}\langle q',t'|q,t\rangle = \phi_0^*(q')\phi_0(q)e^{-E_0|t'|}e^{-E_0|t|}$$

Finally, the Green function can be written as,

$$G(t_1,t_2) = \lim_{\substack{t'\to -i\infty \\ t\to i\infty}}\left[\frac{\langle q',t'|T(Q^H(t_1)Q^H(t_2))|q,t\rangle}{\langle q',t'|q,t\rangle}\right]$$

$$= \lim_{\substack{t'\to -i\infty \\ t\to i\infty}}\frac{1}{\langle q',t'|q,t\rangle}\int\left[\frac{dpdq}{2\pi}\right]q(t_1)q(t_2)\exp\{i\int_t^{t'}d\tau[p\dot{q}-H(p,q)]\}$$

This can generalized to n-point Green's function with the result,

$$G(t_1,t_2,...,t_n) = \langle 0|T(q(t_1)q(t_2)...q(t_n))|0\rangle$$

$$= \lim_{\substack{t' \to -i\infty \\ t \to i\infty}} \frac{1}{\langle q', t' | q, t \rangle} \int \left[\frac{dpdq}{2\pi} \right] q(t_1) q(t_2) \ldots q(t_n) \exp\{ i \int_t^{t'} d\tau [p\dot{q} - H(p,q)] \}$$

It is very useful to introduce generating functional for these n-point functions

$$W[J] = \lim_{\substack{t' \to -i\infty \\ t \to i\infty}} \frac{1}{\langle q', t' | q, t \rangle} \int \left[\frac{dpdq}{2\pi} \right] \exp\{ i \int_t^{t'} d\tau [p\dot{q} - H(p,q) + J(\tau) q(\tau)] \}$$

Then

$$G(t_1, t_2, \ldots, t_n) = (-i)^n \left. \frac{\delta^n}{\delta J(t_1) \ldots \delta J(t_n)} \right|_{J=0}$$

The unphysical limit, $t' \to -i\infty, t \to i\infty$, should be interpreted in term of Eudidean Green's functions defined by

$$S^{(n)}(\tau_1, \tau_2, \ldots, \tau_n) = i^n G^{(n)}(-i\tau_1, -i\tau_2, \ldots, -i\tau_n)$$

Generating functional for $S^{(n)}$ is then

$$W_E[J] = \lim_{\substack{\tau' \to \infty \\ \tau \to -\infty}} \int [dq] \frac{1}{\langle q', t' | q, t \rangle} \exp\{ \int_\tau^{\tau'} d\tau'' [-\frac{m}{2} \left(\frac{dq}{d\tau''} \right)^2 - V(q) + J(\tau'') q(\tau'')] \}$$

Since we can adjust the zero point of $V(q)$ such that

$$\frac{m}{2} \left(\frac{dq}{d\tau} \right)^2 + V(q) > 0$$

which provides the damping to give a converging Gaussian integral. In this form, we can see that any constant in the path integral which is independent of q will be canceled out in the generation functional.

Example : Free particle with mass m moving in one dimension

The Hamiltonian is given by

$$H = \frac{p^2}{2m}$$

The action can be written in terms of the space-time intervals as

$$\begin{aligned} S &= \int_t^{t'} L dt'' = \int_t^{t'} \frac{m}{2} \dot{q}^2 dt'' = \frac{m}{2} \sum_{i=1}^{n-1} \left(\frac{q_i - q_{i+1}}{\Delta} \right)^2 \Delta \\ &= \frac{m}{2\Delta} \left[(q - q_1)^2 + (q_1 - q_2)^2 + \ldots\ldots + (q_{n-1} - q')^2 \right]. \end{aligned} \tag{2}$$

where one has used $n\Delta = (t' - t)$. Using this and the given integration measure, the transition amplitude can be expressed as

$$\begin{aligned} \langle q', t' | q, t \rangle = \left(\frac{m}{2\pi i} \right)^{\frac{n}{2}} \int \prod_{i=1}^{n-1} dq_i \times \\ \exp \left\{ \frac{im}{2\Delta} \left[(q - q_1)^2 + (q_1 - q_2)^2 + \ldots + (q_{n-1} - q')^2 \right] \right\} \end{aligned} \tag{3}$$

The successive integrals can be calculated by using the formulae for Gaussian integrals of the form,

$$\int_{-\infty}^{\infty} dx \exp \left[a (x - x_1)^2 + b (x - x_2)^2 \right] = \sqrt{\frac{-\pi}{a+b}} \exp \left[\frac{ab}{a+b} (x_1 - x_2)^2 \right],$$

so that one has

$$\int dq_1 \exp\left\{\frac{im}{2\Delta}\left[(q-q_1)^2+(q_1-q_2)^2\right]\right\} = \sqrt{\frac{2\pi i\Delta}{m}\cdot\frac{1}{2}}\exp\left[\frac{im}{2\Delta}\frac{(q-q_2)^2}{2}\right]$$

$$\int dq_2 \exp\left\{\frac{im}{2\Delta}\left[\frac{(q-q_2)^2}{2}+(q_2-q_3)^2\right]\right\} = \sqrt{\frac{2\pi i\Delta}{m}\cdot\frac{2}{3}}\exp\left[\frac{im}{2\Delta}\frac{(q-q_3)^2}{3}\right]$$

$$\int dq_3 \exp\left\{\frac{im}{2\Delta}\left[\frac{(q-q_3)^2}{3}+(q_3-q_4)^2\right]\right\} = \sqrt{\frac{2\pi i\Delta}{m}\cdot\frac{3}{4}}\exp\left[\frac{im}{2\Delta}\frac{(q-q_4)^2}{4}\right]$$

so on and so forth. In this way, one obtains

$$\begin{aligned}\left\langle q', t' | q, t\right\rangle &= \lim_{n\to\infty}\left(\frac{m}{2\pi i\Delta}\right)^{\frac{n}{2}}\left(\frac{2\pi i\Delta}{m}\right)^{\frac{n-1}{2}}\left(\frac{1}{2}\,\frac{2}{3}\ldots.\frac{n-1}{n}\right)^{\frac{1}{2}}\exp\left[\frac{im}{2n\Delta}(q-q')^2\right] \\ &= \lim_{n\to\infty}\left(\frac{m}{2\pi in\Delta}\right)^{\frac{1}{2}}\exp\left[\frac{im(q'-q)^2}{2(t'-t)}\right] \\ &= \left(\frac{m}{2\pi i(t'-t)}\right)^{\frac{1}{2}}\exp\left[\frac{im(q'-q)^2}{2(t'-t)}\right]. \end{aligned} \quad (4)$$

In fact this simple case we can compute the transistion amplitude directly. Start with

$$\begin{aligned}\left\langle q', t' | q, t\right\rangle &= \left\langle q' \left| \exp\left[-iH(t'-t)\right]\right| q\right\rangle \quad (5) \\ &= \left\langle q'\right| \exp\left[\frac{-ip^2}{2m}(t'-t)\right]\left|q\right\rangle\end{aligned}$$

Inserting a complete set of momentum states, we get

$$\begin{aligned}\left\langle q', t' | q, t\right\rangle &= \int \frac{dp}{2\pi} \left\langle q'\right| \exp\left[\frac{-ip^2}{2m}\left(t'-t\right)\right] \left|p\right\rangle \left\langle p|q\right\rangle \\ &= \int \frac{dp}{2\pi} \exp\left[\frac{-ip^2}{2m}\left(t'-t\right) + ip\left(q'-q\right)\right], \end{aligned} \tag{6}$$

which can be integrated by using the Gaussian integral formula:

$$\int_{-\infty}^{\infty} dx \exp\left(-ax^2 + bx\right) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right). \tag{7}$$

In our case, we have $a = \frac{i}{2m}\left(t'-t\right)$ and $b = i\left(q'-q\right)$.Thus,

$$\left\langle q', t' | q, t\right\rangle = \sqrt{\frac{m}{2\pi i\left(t'-t\right)}} \exp\left[\frac{im}{2}\frac{\left(q'-q\right)^2}{t'-t}\right]. \tag{8}$$

Field Theory

From quantum mechanics to field theory of a scalar field $\phi(x)$ replace,

$$\prod_{i=1}^{\infty}[dq_i dp_i] \longrightarrow [d\phi(x)d\pi(x)]$$

$$L(q,\dot{q}) \longrightarrow \int \mathcal{L}(\phi,\partial_\mu\phi)d^3x \qquad H(p,q) \longrightarrow \int \mathcal{H}(\phi,\pi)d^3x$$

Generating functional is

$$W[J] \backsim \int [d\phi]\exp\{i\int d^4x[\mathcal{L}(\phi,\partial_\mu\phi)+J(x)\phi(x)]\}$$

functional derivative is defined by

$$\frac{\delta F[\phi(x)]}{\delta\phi(y)} = \lim_{\varepsilon\to 0}\frac{F[\phi(x)+\varepsilon\delta(x-y)]-F[\phi(x)]}{\varepsilon}$$

This is the same as

$$\frac{\delta}{\delta J(x)}J(y) = \delta^4(x-y), \qquad \frac{\delta}{\delta J(x)}\int d^4y J(y)\phi(y) = \phi(x)$$

Then

$$\frac{\delta W[J]}{\delta J(y)} = i\int [d\phi]\phi(y)\exp\{i\int d^4x[\mathcal{L}(\phi,\partial_\mu\phi)+J(x)\phi(x)]\} \tag{9}$$

and

$$\frac{\delta^2 W[J]}{\delta J(y_1)\delta J(y_2)} = (i)^2\int [d\phi]\phi(y_1)\phi(y_2)\exp\{i\int d^4x[\mathcal{L}(\phi,\partial_\mu\phi)+J(x)\phi(x)]\}$$

Consider $\lambda\phi^4$ theory

$$\mathcal{L}(\phi) = \mathcal{L}_0(\phi) + \mathcal{L}_1(\phi)$$

$$\mathcal{L}_0(\phi) = \frac{1}{2}(\partial_\lambda \phi)^2 - \frac{\mu^2}{2}\phi^2, \qquad \mathcal{L}_1(\phi) = -\frac{\lambda}{4!}\phi^4$$

Use Euclidean time the generating functional

$$W[J] = \int [d\phi] \exp\{-\int d^4x[\frac{1}{2}(\frac{\partial\phi}{\partial\tau})^2 + \frac{1}{2}(\overrightarrow{\bigtriangledown}\phi)^2 + \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 - J\phi]\}$$

can be written as

$$W[J] = \left[\exp\int d^4x \mathcal{L}_I\left(\frac{\delta}{\delta J(x)}\right)\right] W_0[J]$$

We have used Eq(9) to write interaction term as functional derivative with repect to the source $J(x)$ .Here $W_0[J]$ is the free field generating functional

$$W_0[J] = \int [d\phi] \exp[-\frac{1}{2}\int d^4x d^4y \phi(x) K(x,y) \phi(y) + \int d^4z J(z)\phi(z)]$$

and

$$K(x,y) = \delta^4(x-y)\left(-\frac{\partial^2}{\partial\tau^2} - \overrightarrow{\bigtriangledown}^2 + \mu^2\right)$$

. The Gaussian integral for many variables is

$$\int d\phi_1 d\phi_2 ... d\phi_n \exp\left[-\frac{1}{2}\sum_{i,j}\phi_i K_{ij}\phi_j + \sum_k J_k\phi_k\right] \backsim \frac{1}{\sqrt{detK}} \exp\left[\frac{1}{2}\sum_{i,j} J_i (K^{-1})_{ij} J_j\right]$$

This can be derived as follows. For Gaussian integral in one variable, we have

$$I = \int_{-\infty}^{\infty} \exp\left(-ax^2 + bx\right) dx = \int_{-\infty}^{\infty} dx \exp[-a\left(x + \frac{b}{2a}\right)^2 + \frac{b^2}{4a}] = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right)$$

Generalization to more than one varible,

$$I_n = \int dx_1 \cdots dx_n \exp[-\frac{1}{2}\sum_{ij} A_{ij} x_i x_j + \sum_j b_j x_j] = \int dx_1 \cdots dx_n \exp[-\frac{1}{2}\left(x, Ax\right) + \left(B, x\right)]$$

where

$$\left(x, Ax\right) = \sum_{ij} A_{ij} x_i x_j, \qquad \left(B, x\right) = \sum_j b_j x_j$$

Since A is a real symmetric matrix, it can be diagonalized by a othorgonal matrix S,

$$SAS^{-1} = D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}, \qquad \text{or} \qquad A = S^{-1}DS = S^T DS$$

Then

$$\left(x, Ax\right) = \left(Sx, DSx\right) = \left(y, Dy\right), \qquad \left(B, x\right) = \left(B', y\right) \qquad \text{where} \quad y = Sx, \quad B' = SB$$

We can then write

$$\begin{aligned} I_n &= \int dy_1 \cdots dy_n \exp[-\frac{1}{2}\left(y, Dy\right) + \left(B', y\right)] = \prod_i \left(\int_{-\infty}^{\infty} \exp\left(-\frac{y_i^2}{2d_i} + b_i' y_i\right) dy_i\right) \\ &= \prod_i \left[\sqrt{\frac{2\pi}{d_i}} \exp\left(\frac{b_i^2}{2d_i}\right)\right] \end{aligned}$$

Note that

$$\prod_i \left[\exp\left(\frac{b_i^2}{2d_i}\right)\right] = \exp\left[\sum_i \left(\frac{b_i'^2}{2d_i}\right)\right], \qquad \prod_i \sqrt{\frac{2\pi}{d_i}} = \frac{(2\pi)^{n/2}}{\sqrt{\det D}} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}$$

We can write

$$\sum_i \left(\frac{b_i'^2}{2d_i}\right) = \frac{1}{2}\left(B', D^{-1}B'\right) = \frac{1}{2}\left(SB, D^{-1}SB\right) = \frac{1}{2}\left(B, A^{-1}B\right)$$

The result is then

$$I_n = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \exp\left[\frac{1}{2}\left(B, A^{-1}B\right)\right] = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \exp\left[\frac{1}{2}\left(b_i \left(A^{-1}\right)_{ij} b_j\right)\right]$$

Apply this to the case of scalar fields,

$$W_0\left[J\right] = \exp\left[\frac{1}{2}\int d^4x d^4y J(x) \triangle(x,y) J(y)\right]$$

where

$$\int d^4y K(x,y) \triangle(y,z) = \delta^4(x-z)$$

$\triangle(x,y)$ can be calculated by Fourier transform to give,

$$\triangle(x,y) = \int \frac{d^4k_E}{(2\pi)^4} \frac{e^{ik_E(x-y)}}{k_E^2 + \mu^2}$$

where $k_E = (ik_0, \overrightarrow{k})$,the Euclidean momentum.

We now give an alternative way to derive the same result. Define

$$\phi(x) = \bar{\phi}(x) + \phi_c(x) \qquad \text{where} \qquad \phi_c(x) = \int \Delta(x,z)\, J(z)\, d^4z$$

then we can write

$$\begin{aligned} S &= -\frac{1}{2}\int d^4x d^4y \phi(x) K(x,y) \phi(y) + \int d^4z J(z)\phi(z) \\ &= -\frac{1}{2}\{\int d^4x d^4y \bar{\phi}(x) K(x,y) \bar{\phi}(y) + \int d^4x d^4y \bar{\phi}(x) K(x,y) \phi_c(y) + \\ &\quad \int d^4x d^4y \phi_c(x) K(x,y) \bar{\phi}(y) + \int d^4x d^4y \phi_c(x) K(x,y) \phi_c(y)\} + \int d^4z J(z)\phi(z) \end{aligned}$$

The first term is

$$\int d^4x d^4y \bar{\phi}(x) K(x,y) \phi_c(y) = \int d^4x \bar{\phi}(x) \int d^4y K(x,y) \int \Delta(y,z)\, J(z)\, d^4z = \int d^4x \bar{\phi}(x) J(x)$$

Similarly

$$\int d^4x d^4y \phi_c(x) K(x,y) \bar{\phi}(y) = \int d^4y J(y)\, \bar{\phi}(y)$$

$$\int d^4x d^4y \phi_c(x) K(x,y) \phi_c(y) = \int d^4x d^4y J(x)\, \Delta(x,y)\, J(y)$$

Put all these together,

$$
\begin{aligned}
S &= -\frac{1}{2}\{\int d^4xd^4y\bar{\phi}(x)K(x,y)\bar{\phi}(y)+\int d^4x\bar{\phi}(x)J(x)+\int d^4yJ(y)\,\bar{\phi}(y) \\
&\quad +\int d^4xd^4yJ(x)\,\Delta(x,y)\,J(y)\}+\int d^4zJ(z)[\bar{\phi}(z)+\phi_c(z)] \\
&= -\frac{1}{2}\int d^4xd^4y\bar{\phi}(x)K(x,y)\bar{\phi}(y)+\frac{1}{2}\int d^4xd^4yJ(x)\,\Delta(x,y)\,J(y)
\end{aligned}
$$

The first term is independent of $J(x)$ and can be dropped. We then get the same result as given in $W_0[J]$. Perturbative expansion in power of λ gives

$$W[J]=W_0[J]\{1+\lambda w_1[J]+\lambda^2 w_2[J]+...\}$$

where

$$w_1=-\frac{1}{4!}W_0^{-1}[J]\{\int d^4x\left[\frac{\delta}{\delta J(x)}\right]^4\}W_0[J]$$

$$w_2=-\frac{1}{2\,(4!)^2}W_0^{-1}[J]\{\int d^4x\left[\frac{\delta}{\delta J(x)}\right]^4\}^2W_0[J]$$

Use explicit form for $W_0[J]$,

$$
\begin{aligned}
W_0[J] &= 1+\frac{1}{2}\int d^4xd^4yJ(x)\triangle(x,y)J(y)+ \\
&\quad \left(\frac{1}{2}\right)^2\frac{1}{2!}\int d^4y_1d^4y_2d^4y_3d^4y_4\,[J(y_1)\triangle(y_1,y_2)J(y_2)J(y_3)\triangle(y_3,y_4)J(y_4)]+...
\end{aligned}
$$

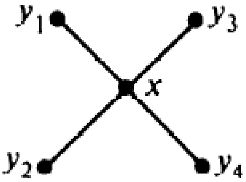
We get for w_1,

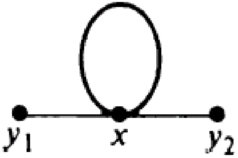
$$w_1 = -\frac{1}{4!}\left[\int \triangle(x,y_1)\triangle(x,y_2)\triangle(x,y_3)\triangle(x,y_4)J(y_1)J(y_2)J(y_3)J(y_4) + 3!\triangle(x,y_1)\triangle(x,y_2)J(y_1)J(y_2)\triangle(x,\right.$$

we dropped all J independent terms, and all (x_i, y_i) are integrated over. In this computation we have used the identity,

$$\frac{\delta}{\delta J(x)}\int d^4y_1 J(y_1) f(y_1) = \int \delta^4(x-y_1)\, d^4y_1 f(y_1) = f(x)$$

Graphical representation for w_1





The connected Green's function is

$$G^{(n)}(x_1, x_2, ...x_n) = \frac{\delta^n \ln W[J]}{\delta J(x_1)\delta J(x_2)...\delta J(x_n)}|_{J=0}$$

Thus replacing y_i by external x_i, we get contributions for 4-point, 2-point functions.

Grassmann algebra

For fermion fields, we need to use anti-commuting c-number functions.This can be realized as elements of Grassmann algebra.

In an n-dimensional Grassmann algebra,the n generators θ_1, θ_2, $\theta_3, ..., \theta_n$ satisfy the anti-commutation relations,

$$\{\theta_i, \theta_j\} = 0 \qquad i, j = 1, 2, ..., n$$

and every element can be expanded in a finite series,

$$P(\theta) = P_0 + P^{(1)}_{i_1}\theta_{i_1} + P^{(2)}_{i_1 i_2}\theta_{i_1}\theta_{i_2} + ... + P_{i_1...i_n}\theta_{i_1}...\theta_{i_n}$$

Simplest case:n=1

$$\{\theta, \theta\} = 0 \qquad \text{or} \qquad \theta^2 = 0 \qquad P(\theta) = P_0 + \theta P_1$$

We can define the "differentiation" and "integration" as follows,

$$\frac{d}{d\theta}\theta = \theta\frac{\overleftarrow{d}}{d\theta} = 1 \quad \Longrightarrow \frac{d}{d\theta}P(\theta) = P_1$$

Integration is defined in such a way that it is invariant under translation,

$$\int d\theta P(\theta) = \int d\theta P(\theta + \alpha)$$

α is another Grassmann variable. This implies

$$\int d\theta = 0$$

We can normalize the integral

$$\int d\theta\theta = 1$$

Then

$$\int d\theta P(\theta) = P_1 = \frac{d}{d\theta} P(\theta)$$

Consider a change of variable

$$\theta \longrightarrow \widetilde{\theta} = a + b\theta$$

Since

$$\int d\widetilde{\theta} P\left(\widetilde{\theta}\right) = \frac{d}{d\widetilde{\theta}} P\left(\widetilde{\theta}\right) = P_1$$

$$\int d\theta P\left(\widetilde{\theta}\right) = \int d\theta \left[P_0 + \widetilde{\theta} P_1\right] = \int d\theta \left[P_0 + (a + b\theta) P_1\right] = bP_1$$

we get

$$\int d\widetilde{\theta} P\left(\widetilde{\theta}\right) = \int d\theta \left(\frac{d\widetilde{\theta}}{d\theta}\right)^{-1} P\left(\widetilde{\theta}(\theta)\right)$$

The "Jacobian" is the inverse of that for c-number integration.

Generalize to n-dimensional Grassmann algebra,

$$\frac{d}{d\theta_i}\left(\theta_1,\theta_2,\theta_3,...,\theta_n\right)=\delta_{i_1}\theta_2...\theta_n-\delta_{i_2}\theta_1\theta_3...\theta_n+...+(-1)^{n-1}\delta_{in}\theta_1\theta_2...\theta_{n-1}$$

$$\left\{d\theta_i,d\theta_j\right\}=0$$

$$\int d\theta_i=0 \qquad \int d\theta_i\theta_j=\delta_{ij}$$

For a change of variables of the form

$$\widetilde{\theta}_i=b_{ij}\theta_j$$

we have

$$\int d\widetilde{\theta}_n d\widetilde{\theta}_{n-1}...d\widetilde{\theta}_1 P\left(\widetilde{\theta}\right)=\int d\theta_n...d\theta_1\left[det\frac{d\widetilde{\theta}}{d\theta}\right]^{-1}P\left(\widetilde{\theta}\left(\theta\right)\right)$$

Proof:

$$\widetilde{\theta}_1\widetilde{\theta}_2...\widetilde{\theta}_n=b_{1i_1}b_{2i_2}...b_{ni_n}\theta_{i_1}...\theta_{i_n}$$

RHS is non-zero only if $i_1,i_2...,i_n$ are all different and we can write

$$\begin{aligned}\widetilde{\theta}_1\widetilde{\theta}_2...\widetilde{\theta}_n &= b_{1i_1}b_{2i_2}...b_{ni_n}\epsilon_{i_1,i_2...,i_n}\theta_{i_1}...\theta_{i_n}\\ &= (\det b)\,\theta_1\theta_2\theta_3...\theta_n\end{aligned}$$

From the normalization condition,

$$1=\int d\widetilde{\theta}_n d\widetilde{\theta}_{n-1}...d\widetilde{\theta}_1\left(\widetilde{\theta}_1\widetilde{\theta}_2...\widetilde{\theta}_n\right)=(\det b)\int d\widetilde{\theta}_n d\widetilde{\theta}_{n-1}...d\widetilde{\theta}_1\left(\theta_1\theta_2\theta_3...\theta_n\right)$$

we see that

$$d\widetilde{\theta}_n d\widetilde{\theta}_{n-1}...d\widetilde{\theta}_1=(\det b)^{-1}d\theta_1...d\theta_n$$

In field theory, we need Gaussian integral of the form,

$$G(A) \equiv \int d\theta_n ... d\theta_1 \exp\left(\frac{1}{2}(\theta, A\theta)\right) \qquad \text{where } (\theta, A\theta) = \theta_i A_{ij} \theta_j$$

First consider n=2

$$A = \begin{pmatrix} 0 & A_{12} \\ -A_{12} & 0 \end{pmatrix}$$

Then

$$G(A) = \int d\theta_2 d\theta_1 \exp(\theta_1 \theta_2 A_{12}) \simeq \int d\theta_2 d\theta_1 (1 + \theta_1 \theta_2 A_{12}) = A_{12} = \sqrt{\det A}$$

For the general $n = even$, we first bring the matrix A into the standard form by a unitary transformation,

$$UAU^{\dagger} = A_s$$

$$A_s = \begin{bmatrix} a\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \\ & & \ddots \end{bmatrix}$$

This can be seen as follows. Since iA is Hermitian, it cab diagonalized by a unitary transfomation,

$$V(iA)V^{\dagger} = A_d$$

where A_d is real and diagonal. The diagonal elements are solutions to the secular equation,

$$\det|iA - \lambda I| = 0$$

Since $A = -A^T$, we have

$$0 = \det|iA - \lambda I|^T = \det|-iA - \lambda I|$$

This means that if λ is a solution, -λ is also a solution and A_d is of the form,

$$A_d = \begin{pmatrix} a & & & & \\ & -a & & & \\ & & b & & \\ & & & -b & \\ & & & & \ddots \end{pmatrix}$$

To put this matrix into the standard we use the unitary matrix

$$S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$$

which has the property

$$S_2 (-i) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S_2^\dagger = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Thus we get

$$S(-iA_d)S^\dagger = A_s, \quad \text{where} \quad S = \begin{pmatrix} S_2 & & \\ & S_2 & \\ & & \ddots \end{pmatrix}$$

For arbitrary n, we get

$$G(A) = \int d\theta_n ... d\theta_1 \exp\left(\frac{1}{2}(\theta, A\theta)\right) = \sqrt{\det A} \qquad \text{n even}$$

and for "complex" Grassmann variables

$$\int d\theta_n d\bar{\theta}_n d\theta_{n-1} d\bar{\theta}_{n-1} ... d\theta_1 d\bar{\theta}_1 \exp\left(\bar{\theta}, A\theta\right) = \det A$$

For the Fermion fields, the generating functional is of the form,

$$W[\eta, \overline{\eta}] = \int [d\psi(x)] \left[d\overline{\psi}(x)\right] \exp\{i \int d^4x \left[\mathcal{L}\left(\psi, \overline{\psi}\right) + \overline{\psi}\eta + \overline{\eta}\psi\right]\}$$

If $\mathcal{L}$ depends on $\psi, \overline{\psi}$ quadratically

$$\mathcal{L} = \left(\overline{\psi}, A\psi\right)$$

then we have

$$W = \int [d\psi(x)] \left[d\overline{\psi}(x)\right] \exp\{\int d^4x \overline{\psi} A\psi\} = \det A$$