Quantum Field Theory

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Path integral formalism has close relationship to classical dynamics, e.g. the transition amplitude

$$\langle f|i\rangle = \int [dx]\, e^{iS/\hbar}$$

as $\hbar \rightarrow 0$, the trajectory with smallest S dominates, the action principle. Here uses the ordinary functions not the operators. Later in non-Abelian gauge theory, to remove unphysical degrees of freedom can be accomodated in the path integral formalism by imposing constraints in the integral.

Quantum Mechanics in 1-dimension

In QM, transition from $|q,t\rangle$ to $\langle q',t'|$, is,

$$\langle q't'|qt\rangle = \langle q'|e^{-iH(t-t')}|q\rangle$$

where $|q\rangle's$ are eigenstates of position operator Q in the Schrodinger picture,

$$Q|q\rangle = q|q\rangle$$

and $|q,t\rangle$ denotes corresponding state in Heisenberg picture,

$$|q,t\rangle = e^{iHt}|q\rangle$$

In path integral formalism, this can be written as

$$\langle q' t' | q t \rangle = N \int [dq] \exp\{ i \int_t^{t'} d\tau L(q, \dot{q}) \}$$

To get this, divide $(t\prime, t)$ into n intervals ,

$$\delta t = \frac{t' - t}{n}$$

and write ,

$$\langle q' | e^{-iH(t'-t)} | q \rangle = \int dq_1 ... dq_{n-1} \langle q' | e^{-iH\delta t} | q_{n-1} \rangle \langle q_{n-1} | e^{-iH\delta t} | q_{n-2} \rangle ... \langle q_1 | e^{-iH\delta t} | q \rangle$$

If we take Hamiltonian to be in the simple form,

$$H(P, Q) = \frac{p^2}{2m} + V(Q)$$

then

$$\begin{aligned} \langle q_j | H | q_i \rangle &= \langle q_j | \frac{p^2}{2m} | q_i \rangle + V(\frac{q_i + q_j}{2}) \delta(q_i - q_j) \\ &= \int \langle q_j | \frac{p^2}{2m} | p_k \rangle \langle p_k | q_i \rangle (\frac{dp_k}{2\pi}) + V(\frac{q_i + q_j}{2}) \int \frac{dp_k}{2\pi} e^{ip_k (q_j - q_i)} \\ &= \int \frac{dp_k}{2\pi} e^{ip_k (q_j - q_i)} [\frac{p_k^2}{2m} + V(\frac{q_i + q_j}{2})] \end{aligned}$$

where we have used

$$\langle p|q\rangle = e^{-ipq}$$

the momentum eigenfunction in coordinate space. Exponentiation of this infinitesmal result gives

$$\langle q_j|e^{-iH\delta t}|q_i\rangle \simeq \int \frac{dp_k}{2\pi} e^{ip_k(q_j-q_i)}\{1-i\delta t[\frac{p_k^2}{2m}+V(\frac{q_i+q_j}{2})]\} \simeq \int \frac{dp_k}{2\pi} e^{ip_k(q_j-q_i)} \exp\{-i\delta t[\frac{p_k^2}{2m}+V(\frac{q_i+q_j}{2})]\}$$

The whole transition matrix element can then be written as

$$\langle q'|e^{-iH(t'-t)}|q\rangle \cong \int (\frac{dp_1}{2\pi})...(\frac{dp_n}{2\pi}) \int dq_1...dq_{n-1} \exp\{i\left[\sum_{i=1}^{n} p_i(q_i-q_{i-1})-(\delta t)H(p_i,\frac{q_i+q_{i+1}}{2})\right]\}$$

This can be written formally as

$$\langle q'|e^{-iH(t'-t)}|q\rangle = \int [\frac{dpdq}{2\pi}] \exp\{i\int_t^{t'} dt[p\dot{q}-H(p,q)]\}$$

$$\equiv \lim_{n\to\infty} \int (\frac{dp_1}{2\pi})...(\frac{dp_n}{2\pi}) \int dq_1...dq_{n-1} \exp\{i\sum_{i=1}\delta t[p_i(\frac{q_i-q_{i-1}}{\delta t})-H(p_i,\frac{q_i+q_{i+1}}{2})]\}$$

If Hamiltonian depends quadractically on p, use the formula

$$\int_{-\infty}^{+\infty} \frac{dx}{2\pi} e^{-ax^2+bx} = \frac{1}{\sqrt{4\pi a}} e^{\frac{b^2}{4a}}$$

to get

$$\int \frac{dp_i}{2\pi} \exp[\frac{-i\delta t}{2m}p_i^2+ip_i(q_i-q_{i-1})] = (\frac{m}{2\pi i\delta t})^{1/2} \exp[\frac{im(q_i-q_{i-1})^2}{2\delta t}]$$

Then

$$\langle q'|e^{-iH(t'-t)}|q\rangle = \lim_{n\to\infty}\left(\frac{m}{2\pi i\delta t}\right)^{n/2}\int\prod_{i=1}^{n-1}dq_i\exp\{i\sum_{i=1}^{n}\delta t[\frac{m}{2}(\frac{q_i - q_{i-1}}{\delta t})^2 - V]\}$$

or

$$\langle q't'|qt\rangle = \langle q'|e^{-iH(t'-t)}|q\rangle = N\int[dq]\exp\{i\int_t^{t'}d\tau[\frac{m}{2}\dot{q}^2 - V(q)]\}$$

This is the path integral representation for amplitude from initial state $|q,t\rangle$ to final state $\langle q',t'|$. Or

$$\langle q't'|qt\rangle = N\int[dq]\exp iS$$

Green's functions

To generalize this to field theory where we have vacuum expectation value of field operators, consider

$$G(t_1, t_2) = \langle 0|T(Q^H(t_1)Q^H(t_2))|0\rangle$$

Inserting complete sets of states,

$$G(t_1, t_2) = \int dqdq' \langle 0|q', t'\rangle\langle q', t'|T(Q^H(t_1)Q^H(t_2))|q, t\rangle\langle q, t|0\rangle$$

The matrix element

$$\langle 0|q, t\rangle = \phi_0(q)e^{-iE_0 t} = \phi_0(q, t)$$

is the wavefunction for ground state. Consider the case

$$t' > t_1 > t_2 > t,$$

we can write

$$\langle q', t'|T(Q^H(t_1)Q^H(t_2))|q, t\rangle = \langle q'|e^{-iH(t'-t_1)}Q^s e^{-iH(t_1-t_2)}Q^s e^{-iH(t_2-t)}|q\rangle$$
$$= \int \langle q'|e^{-iH(t'-t_1)}|q_1\rangle q_1\langle q_1|e^{-iH(t_1-t_2)}|q_2\rangle q_2\langle q_2|e^{-iH(t_2-t)}|q\rangle dq_1 dq_2$$
$$= \int [\frac{dpdq}{2\pi}]q_1(t_1)q_2(t_2)exp\{i\int_t^{t'} d\tau[p\dot{q} - H(p, q)]\}$$

For the other time sequence

$$t' > t_2 > t_1 > t,$$

we get same formula, because path integral orders the time sequence automatically through the division of time interval into small pieces. The Green's function is then

$$G(t_1, t_2) = \int dqdq'\phi_0(q', t')\phi_0^*(q, t)\int[\frac{dpdq}{2\pi}]q_1(t_1)q_2(t_2)exp\{i\int_t^{t'} d\tau[p\dot{q} - H(p, q)]\} \quad (1)$$

We remove wavefunction $\phi_0(q,t)$ as follows. Write

$$\langle q',t'|\theta(t_1,t_2)|q,t\rangle = \int dQdQ'\langle q',t'|Q',T'\rangle\langle Q',T'|\theta(t_1,t_2)|Q,T\rangle\langle Q,t|q,t\rangle$$

where

$$\theta(t_1,t_2) = T(Q^H(t_1)Q^H(t_2))$$

Let $|n>$ be eigenstate with energy E_n and wave function ϕ_n, i.e.,

$$H|n>= E_n|n>, \quad \langle q|n\rangle = \phi_n^*(q)$$

Then

$$\langle q',t'|Q',t'\rangle = \langle q'|e^{-iH(t'-T')}|Q'\rangle = \sum_n\langle q'|n\rangle e^{-iE_n(t'-T')}\langle n|Q'\rangle = \sum_n \phi_n^*(q')\phi_n(Q')e^{-iE_n(t'-T')}$$

To isolate the ground state wavefunction, take an "unusual limit",

$$\lim_{t'\to -i\infty}\langle q',t'|Q',T'\rangle = \phi_0^*(q')\phi_0(Q')e^{-E_0|t'|}e^{iE_0T'}$$

Similarly,

$$\lim_{t\to i\infty}\langle Q,T|q,t\rangle = \phi_0(q)\phi_0^*(Q)e^{-E_0|t|}e^{-iE_0T}$$

With these we write

$$\underset{\substack{t'\to -i\infty \\ t\to i\infty}}{lim}\langle q',t'|\theta(t_1,t_2)|q,t\rangle = \int dQdQ'\phi_0^*(q')\phi_0(Q')\langle Q',T'|\theta(t_1,t_2)|Q,T\rangle\phi_0^*(Q)\phi_0(q)e^{-E_0|t'|}e^{iE_0T'}e^{-iE_0T}e^{-E_0|t|}$$

$$= \phi_0^*(q')\phi_0(q)e^{-E_0|t'|}e^{-E_0|t|}G(t_1, t_2)$$

It is easy to see that

$$\lim_{\substack{t'\to -i\infty \\ t\to i\infty}} \langle q', t'|q, t\rangle = \phi_0^*(q')\phi_0(q)e^{-E_0|t'|}e^{-E_0|t|}$$

Finally, the Green function can be written as,

$$G(t_1, t_2) = \lim_{\substack{t'\to -i\infty \\ t\to i\infty}} \left[\frac{\langle q', t'|T(Q^H(t_1)Q^H(t_2))|q, t\rangle}{\langle q', t'|q, t\rangle} \right]$$

$$= \lim_{\substack{t'\to -i\infty \\ t\to i\infty}} \frac{1}{\langle q', t'|q, t\rangle} \int \left[\frac{dpdq}{2\pi} \right] q(t_1)q(t_2) \exp\{i \int_t^{t'} d\tau [p\dot{q} - H(p, q)]\}$$

This can generalized to n-point Green's function with the result,

$$G(t_1, t_2, ..., t_n) = \langle 0|T(q(t_1)q(t_2)...q(t_n))|0\rangle$$

$$= \lim_{\substack{t'\to -i\infty \\ t\to i\infty}} \frac{1}{\langle q', t'|q, t\rangle} \int \left[\frac{dpdq}{2\pi} \right] q(t_1)q(t_2)...q(t_n) \exp\{i \int_t^{t'} d\tau [p\dot{q} - H(p, q)]\}$$

It is very useful to introduce generating functional for these n-point functions

$$W[J] = \lim_{\substack{t' \to -i\infty \\ t \to i\infty}} \frac{1}{\langle q', t' | q, t \rangle} \int \left[\frac{dpdq}{2\pi} \right] \exp\{ i \int_t^{t'} d\tau [p\dot{q} - H(p,q) + J(\tau) q(\tau)] \}$$

Then

$$G(t_1, t_2, ..., t_n) = (-i)^n \left. \frac{\delta^n}{\delta J(t_1) ... \delta J(t_n)} \right|_{J=0}$$

The unphysical limit, $t' \to -i\infty, t \to i\infty$, should be interpreted in term of Eudidean Green's functions defined by

$$S^{(n)}(\tau_1, \tau_2, ..., \tau_n) = i^n G^{(n)}(-i\tau_1, -i\tau_2, ..., -i\tau_n)$$

Generating functional for $S^{(n)}$ is then

$$W_E[J] = \lim_{\substack{\tau' \to \infty \\ \tau \to -\infty}} \int [dq] \frac{1}{\langle q', t' | q, t \rangle} \exp\{ \int_\tau^{\tau'} d\tau'' [-\frac{m}{2} \left(\frac{dq}{d\tau''} \right)^2 - V(q) + J(\tau'') q(\tau'')] \}$$

Since we can adjust the zero point of $V(q)$ such that

$$\frac{m}{2} \left(\frac{dq}{d\tau} \right)^2 + V(q) > 0$$

which provides the damping to give a converging Gaussian integral. In this form, we can see that any constant in the path integral which is independent of q will be canceled out in the generation functional.

Example : Free particle with mass m moving in 1- dimension
The Hamiltonian is given by

$$H = \frac{p^2}{2m}$$

The action can be written in terms of the space-time intervals as

$$\begin{aligned} S &= \int_t^{t'} L dt'' = \int_t^{t'} \frac{m}{2} \dot{q}^2 dt'' = \frac{m}{2} \sum_{i=1}^{n-1} \left(\frac{q_i - q_{i+1}}{\Delta} \right)^2 \Delta \\ &= \frac{m}{2\Delta} \left[(q - q_1)^2 + (q_1 - q_2)^2 + \ldots\ldots + (q_{n-1} - q')^2 \right]. \end{aligned} \tag{2}$$

where one has used $n\Delta = (t' - t)$. The transition amplitude can be expressed as

$$\langle q', t' | q, t \rangle = \left(\frac{m}{2\pi i} \right)^{\frac{n}{2}} \int \prod_{i=1}^{n-1} dq_i \times \exp \left\{ \frac{im}{2\Delta} \left[(q - q_1)^2 + (q_1 - q_2)^2 + \ldots + (q_{n-1} - q')^2 \right] \right\} \tag{3}$$

The successive integrals can be calculated by using the formulae for Gaussian integrals of the form,

$$\int_{-\infty}^{\infty} dx \exp \left[a (x - x_1)^2 + b (x - x_2)^2 \right] = \sqrt{\frac{-\pi}{a + b}} \exp \left[\frac{ab}{a + b} (x_1 - x_2)^2 \right],$$

so that

$$\int dq_1 \exp\left\{\frac{im}{2\Delta}\left[(q-q_1)^2+(q_1-q_2)^2\right]\right\} = \sqrt{\frac{2\pi i\Delta}{m}\cdot\frac{1}{2}}\exp\left[\frac{im}{2\Delta}\frac{(q-q_2)^2}{2}\right]$$

$$\int dq_2 \exp\left\{\frac{im}{2\Delta}\left[\frac{(q-q_2)^2}{2}+(q_2-q_3)^2\right]\right\} = \sqrt{\frac{2\pi i\Delta}{m}\cdot\frac{2}{3}}\exp\left[\frac{im}{2\Delta}\frac{(q-q_3)^2}{3}\right]$$

$$\int dq_3 \exp\left\{\frac{im}{2\Delta}\left[\frac{(q-q_3)^2}{3}+(q_3-q_4)^2\right]\right\} = \sqrt{\frac{2\pi i\Delta}{m}\cdot\frac{3}{4}}\exp\left[\frac{im}{2\Delta}\frac{(q-q_4)^2}{4}\right]$$

so on and so forth. In this way, one obtains

$$\begin{aligned}\langle q',t'|q,t\rangle &= \lim_{n\to\infty}\left(\frac{m}{2\pi i\Delta}\right)^{\frac{n}{2}}\left(\frac{2\pi i\Delta}{m}\right)^{\frac{n-1}{2}}\left(\frac{1}{2}\frac{2}{3}\dots\frac{n-1}{n}\right)^{\frac{1}{2}}\exp\left[\frac{im}{2n\Delta}(q-q')^2\right]\\ &= \lim_{n\to\infty}\left(\frac{m}{2\pi in\Delta}\right)^{\frac{1}{2}}\exp\left[\frac{im(q'-q)^2}{2(t'-t)}\right]\\ &= \left(\frac{m}{2\pi i(t'-t)}\right)^{\frac{1}{2}}\exp\left[\frac{im(q'-q)^2}{2(t'-t)}\right].\end{aligned} \tag{4}$$

In fact this simple case we can compute the transistion amplitude directly. Start with

$$\begin{aligned}\langle q',t'|q,t\rangle &= \langle q'\left|\exp\left[-iH(t'-t)\right]\right|q\rangle \\ &= \langle q'|\exp\left[\frac{-ip^2}{2m}(t'-t)\right]|q\rangle\end{aligned} \tag{5}$$

Inserting a complete set of momentum states, we get

$$\begin{aligned}\left\langle q', t' | q, t\right\rangle &= \int \frac{dp}{2\pi} \left\langle q'\right| \exp\left[\frac{-ip^2}{2m}\left(t'-t\right)\right] \left|p\right\rangle \left\langle p|q\right\rangle \\ &= \int \frac{dp}{2\pi} \exp\left[\frac{-ip^2}{2m}\left(t'-t\right) + ip\left(q'-q\right)\right], \end{aligned} \tag{6}$$

which can be integrated by using the Gaussian integral formula:

$$\int_{-\infty}^{\infty} dx \exp\left(-ax^2 + bx\right) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right). \tag{7}$$

In our case, we have $a = \frac{i}{2m}\left(t'-t\right)$ and $b = i\left(q'-q\right)$ .Thus,

$$\left\langle q', t' | q, t\right\rangle = \sqrt{\frac{m}{2\pi i\left(t'-t\right)}} \exp\left[\frac{im}{2}\frac{\left(q'-q\right)^2}{t'-t}\right]. \tag{8}$$

Field Theory

From quantum mechanics to field theory of a scalar field $\phi(x)$ replace,

$$\prod_{i=1}^{\infty}[dq_i dp_i] \longrightarrow [d\phi(x) d\pi(x)]$$

$$L(q,\dot{q}) \longrightarrow \int \mathcal{L}(\phi, \partial_\mu \phi) d^3x \qquad H(p,q) \longrightarrow \int \mathcal{H}(\phi, \pi) d^3x$$

Generating functional is

$$W[J] \backsim \int [d\phi] \exp\{i \int d^4x [\mathcal{L}(\phi, \partial_\mu \phi) + J(x)\phi(x)]\}$$

functional derivative is defined by

$$\frac{\delta F[\phi(x)]}{\delta \phi(y)} = \lim_{\varepsilon \to 0} \frac{F[\phi(x) + \varepsilon \delta(x-y)] - F[\phi(x)]}{\varepsilon}$$

This is the same as

$$\frac{\delta}{\delta J(x)} J(y) = \delta^4(x-y), \qquad \frac{\delta}{\delta J(x)} \int d^4y J(y)\phi(y) = \phi(x)$$

Then

$$\frac{\delta W[J]}{\delta J(y)} = i \int [d\phi] \phi(y) \exp\{i \int d^4x [\mathcal{L}(\phi, \partial_\mu \phi) + J(x)\phi(x)]\} \tag{9}$$

and

$$\frac{\delta^2 W[J]}{\delta J(y_1) \delta J(y_2)} = (i)^2 \int [d\phi] \phi(y_1) \phi(y_2) \exp\{i \int d^4x [\mathcal{L}(\phi, \partial_\mu \phi) + J(x)\phi(x)]\}$$

The Green's functions can be written in terms of the generating functionals as

$$G^{(n)}\left(x_1, x_2, \cdots x_n\right) = \left\langle 0 \,|\, T\left(\phi\left(x_1\right)\phi\left(x_2\right)\cdots\phi\left(x_n\right)\right) |\, 0\right\rangle = \left.\frac{\delta^{(n)}\ Z}{\delta J\left(x_1\right)\cdots\delta J\left(x_n\right)}\right|_{J=0} \tag{10}$$

$$= \left[\frac{\delta^{(n)}}{\delta J\left(x_1\right)\cdots\delta J\left(x_n\right)} \exp\{\int d^4 y[\mathcal{L}(\phi, \partial_\mu \phi) + J(y)\phi(y)]\}\right]_{J=0}$$

Connected Green's functions

The Green's functions in Eq (10) contain both connected and disconnected Green's functions. Only the connected Green's functions are of physical interest to and can be obtained from taking functional derivatives of,

$$W[J] = -i \ln Z[J] \tag{11}$$

We now demonstrate this in the simple case of free field theory. As we will see later the free field generating functional is of the form

$$Z_0[J] = exp\left[\frac{1}{2}\int d^4x_1 d^4x_2 J(x_1) \triangle (x_1, x_2) J(x_2)\right] \tag{12}$$

Expand this exponential, we find that

$$Z_0[J] = \left[\begin{array}{c} 1 + \frac{1}{2}\int d^4x_1 d^4x_2 J(x_1)\triangle(x_1,x_2)J(x_2) \\ +\frac{1}{2!}\left(\frac{1}{2}\right)^2 \int d^4x_1 d^4y_1 J(x_1)\triangle(x_1,y_1)J(y_1)\int d^4x_2 d^4y_2 J(x_2)\triangle(x_2,y_2)J(y_2) + \cdots \end{array}\right]$$

We denote this symbolically by the following graph,

$$Z_0 = 1 + \frac{1}{2}(J - -J) + \left(\frac{1}{2}\right)^2 \frac{1}{2!}(J - -J)(J - -J) + \cdots$$

Note that here source function $J(x)$ always occurs in pair connected by $\triangle(x, y)$. So when we differentiate with respect to $J(x)$ to get the Green's functions, the differentiation on 2 sources funtions $J(x)$ and $J(y)$ connected by $\triangle(x, y)$ will correspond to a contraction in Wick's Theorem. When we differentiate with respect to the source function $J(x)$ to get the Green's functions, we see that

$$\left.\frac{\delta Z_0[J]}{\delta J(y)}\right|_{J=0} = 0, \qquad \text{and all odd order differentiations vanish}$$

For the 2-point function we have

$$\langle 0 \,|\, T\,(\phi\,(y_1)\,\phi\,(y_2))|\,0\rangle = \left.\frac{\delta^2 Z\,[J]}{\delta J\,(y_1)\,\delta J\,(y_2)}\right|_{J=0} = \triangle(y_1, y_2)$$

This corresponds to removing 2 sources $J(x_1), J(x_2)$ from the second term in Z_0 and left with a propagator

$$y_1 - - - - y_2$$

Similarly,

$$\langle 0 \,|\, T\,(\phi\,(y_1)\,\phi\,(y_2)\,\phi\,(y_3)\,\phi\,(y_4))|\,0\rangle = \left.\frac{\delta^4 Z\,[J]}{\delta J\,(y_1)\,\delta J\,(y_2)\,\delta J\,(y_3)\,\delta J\,(y_4)}\right|_{j=0} = \triangle(y_1, y_2)\,\triangle\,(y_3, y_4) + permutatios$$

$$\begin{array}{c} y_1 - - - y_2 \\ y_3 - - - y_4 \end{array} \quad + permutations$$

Thus the 4-point function consists of 2 disconnected 2 point functions. From this we can see that higher point functions are also made out of disconnected graphs. This is what we expected because without interaction we only get disconnected graphs. But when we take the logarithm of Z_0, we get

$$W_0 = -i \ln Z_0 = -\frac{1}{2} \int d^4x d^4y J(x) \triangle (x, y) J(y)$$

Here we have only 2-point function and all disconnected parts dropped out.

Free Generating functional

We now discuss the perturbation expansion. Consider $\lambda\phi^4$ theory

$$\mathcal{L}(\phi) = \mathcal{L}_0(\phi) + \mathcal{L}_1(\phi)$$

$$\mathcal{L}_0(\phi) = \frac{1}{2}(\partial_\lambda \phi)^2 - \frac{\mu^2}{2}\phi^2, \qquad \mathcal{L}_1(\phi) = -\frac{\lambda}{4!}\phi^4$$

Using Euclidean time, the generating functional

$$W[J] = \int [d\phi] \exp\{-\int d^4x [\frac{1}{2}(\frac{\partial\phi}{\partial\tau})^2 + \frac{1}{2}(\overrightarrow{\bigtriangledown}\phi)^2 + \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 - J\phi]\}$$

can be written as

$$W[J] = \left[\exp \int d^4x \mathcal{L}_I \left(\frac{\delta}{\delta J(x)}\right)\right] W_0[J]$$

We have used Eq(9) to write interaction term as functional derivative with repect to the source $J(x)$ .Here $W_0[J]$ is the free field generating functional

$$W_0[J] = \int [d\phi] \exp[-\frac{1}{2}\int d^4x d^4y \phi(x) K(x,y) \phi(y) + \int d^4z J(z) \phi(z)]$$

and

$$K(x,y) = \delta^4(x-y)\left(-\frac{\partial^2}{\partial\tau^2} - \overrightarrow{\bigtriangledown}^2 + \mu^2\right)$$

.

The Gaussian integral for many variables is

$$\int d\phi_1 d\phi_2 ... d\phi_n \exp\left[-\frac{1}{2}\sum_{i,j}\phi_i K_{ij}\phi_j + \sum_k J_k\phi_k\right] \backsim \frac{1}{\sqrt{detK}} \exp\left[\frac{1}{2}\sum_{i,j} J_i (K^{-1})_{ij} J_j\right]$$

This can be derived as follows. For Gaussian integral in one variable, we have

$$I = \int_{-\infty}^{\infty} \exp\left(-ax^2 + bx\right) dx = \int_{-\infty}^{\infty} dx \exp[-a\left(x + \frac{b}{2a}\right)^2 + \frac{b^2}{4a}] = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right)$$

Generalization to more than one varible,

$$I_n = \int dx_1 \cdots dx_n \exp[-\frac{1}{2}\sum_{ij} A_{ij}x_i x_j + \sum_j b_j x_j] = \int dx_1 \cdots dx_n \exp[-\frac{1}{2}(x, Ax) + (B, x)]$$

where

$$(x, Ax) = \sum_{ij} A_{ij} x_i x_j, \qquad (B, x) = \sum_j b_j x_j$$

Since A is a real symmetric matrix, it can be diagonalized by a othorgonal matrix S,

$$SAS^{-1} = D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}, \qquad \text{or} \qquad A = S^{-1}DS = S^T DS$$

Then

$$(x, Ax) = (Sx, DSx) = (y, Dy), \qquad (B, x) = (B', y) \qquad \text{where} \quad y = Sx, \quad B' = SB$$

We can then write

$$\begin{aligned} I_n &= \int dy_1 \cdots dy_n \exp[-\frac{1}{2}(y, Dy) + (B', y)] = \prod_i \left(\int_{-\infty}^{\infty} \exp\left(-\frac{y_i^2}{2d_i} + b_i' y_i \right) dy_i \right) \\ &= \prod_i \left[\sqrt{\frac{2\pi}{d_i}} \exp\left(\frac{b_i^2}{2d_i} \right) \right] \end{aligned}$$

Note that

$$\prod_i \left[\exp\left(\frac{b_i^2}{2d_i} \right) \right] = \exp\left[\sum_i \left(\frac{b_i'^2}{2d_i} \right) \right], \qquad \prod_i \sqrt{\frac{2\pi}{d_i}} = \frac{(2\pi)^{n/2}}{\sqrt{\det D}} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}$$

We can write

$$\sum_i \left(\frac{b_i'^2}{2d_i} \right) = \frac{1}{2}(B', D^{-1}B') = \frac{1}{2}(SB, D^{-1}SB) = \frac{1}{2}(B, A^{-1}B)$$

The result is then

$$I_n = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \exp\left[\frac{1}{2}(B, A^{-1}B) \right] = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \exp\left[\frac{1}{2}\left(b_i \left(A^{-1} \right)_{ij} b_j \right) \right]$$

Apply this to the case of scalar fields,

$$W_0[J] = \exp\left[\frac{1}{2} \int d^4x d^4y J(x) \triangle(x, y) J(y) \right]$$

where

$$\int d^4y K(x, y) \triangle(y, z) = \delta^4(x - z)$$

$\triangle(x,y)$ can be calculated by Fourier transform to give,

$$\triangle(x,y) = \int \frac{d^4k_E}{(2\pi)^4}\frac{e^{ik_E(x-y)}}{k_E^2+\mu^2}$$

where $k_E = (ik_0, \overrightarrow{k})$,the Euclidean momentum.
We now give an alternative way to derive the same result. Define

$$\phi(x) = \bar{\phi}(x) + \phi_c(x) \qquad \text{where} \qquad \phi_c(x) = \int \Delta(x,z)\, J(z)\, d^4z$$

then we can write

$$\begin{aligned} S &= -\frac{1}{2}\int d^4xd^4y\phi(x)K(x,y)\phi(y) + \int d^4zJ(z)\phi(z) \\ &= -\frac{1}{2}\{\int d^4xd^4y\bar{\phi}(x)K(x,y)\bar{\phi}(y) + \int d^4xd^4y\bar{\phi}(x)K(x,y)\phi_c(y) + \\ &\quad \int d^4xd^4y\phi_c(x)K(x,y)\bar{\phi}(y) + \int d^4xd^4y\phi_c(x)K(x,y)\phi_c(y)\} + \int d^4zJ(z)\phi(z) \end{aligned}$$

The first term is

$$\int d^4xd^4y\bar{\phi}(x)K(x,y)\phi_c(y) = \int d^4x\bar{\phi}(x)\int d^4yK(x,y)\int \Delta(y,z)\, J(z)\, d^4z = \int d^4x\bar{\phi}(x)J(x)$$

Similarly

$$\int d^4xd^4y\phi_c(x)K(x,y)\bar{\phi}(y) = \int d^4yJ(y)\,\bar{\phi}(y)$$

$$\int d^4xd^4y\phi_c(x)K(x,y)\phi_c(y) = \int d^4xd^4yJ(x)\,\Delta(x,y)\,J(y)$$

Put all these together,

$$
\begin{aligned}
S &= -\frac{1}{2}\{\int d^4xd^4y\bar{\phi}(x)K(x,y)\bar{\phi}(y) + \int d^4x\bar{\phi}(x)J(x) + \int d^4yJ(y)\bar{\phi}(y) \\
&\quad + \int d^4xd^4yJ(x)\Delta(x,y)J(y)\} + \int d^4zJ(z)[\bar{\phi}(z) + \phi_c(z)] \\
&= -\frac{1}{2}\int d^4xd^4y\bar{\phi}(x)K(x,y)\bar{\phi}(y) + \frac{1}{2}\int d^4xd^4yJ(x)\Delta(x,y)J(y)
\end{aligned}
$$

The first term is independent of $J(x)$ and can be dropped. We then get the same result as given in $W_0[J]$.

Perturbation Expansion and Feynman Diagrams

Back to the full generating functional $Z[J]$ which is related to the free generating functional $Z_0[J]$ by,

$$Z[J] = \left[\exp \int d^4x \mathcal{L}_I \left(\frac{\delta}{\delta J(x)}\right)\right] Z_0[J] = \left[\exp \int d^4x \frac{\lambda}{4!} \left(\frac{\delta}{\delta J(x)}\right)^4\right] Z_0[J]$$

Perturbative expansion in power of λ gives

$$Z[J] = \left[1 + \frac{\lambda}{4!} \int d^4x \left(\frac{\delta}{\delta J(x)}\right)^4 + \left(\frac{\lambda}{4!}\right)^2 \left(\int d^4x \left(\frac{\delta}{\delta J(x)}\right)^4\right)^2 + \cdots\right] Z_0[J]$$

Or

$$Z[J] = Z_0[J]\{1 + \lambda z_1[J] + + ...\}$$

where

$$w_1 = \frac{1}{4!} \frac{1}{Z_0[J]} \{\int d^4x \left[\frac{\delta}{\delta J(x)}\right]^4\} Z_0[J]$$

To compute w_1,which involve functional differentiation with respect to $J(x)$, we first expand $Z_0[J]$ in powers of $J(x)$

$$\begin{aligned} Z_0[J] &= 1 + \frac{1}{2} \int d^4y_1 d^4y_2 J(y_1) \triangle (y_1, y_2) J(y_2) \\ &+ \left(\frac{1}{2}\right)^2 \frac{1}{2!} \int d^4y_1 d^4y_2 d^4y_3 d^4y_4 \left[J(y_1) \triangle (y_1, y_2) J(y_2) J(y_3) \triangle (y_3, y_4) J(y_4)\right] + \ldots \end{aligned}$$

It is clear that in Z_0 only even powers of J show up. For the sake of notation we write this as

$$Z_0\left[J\right] = 1 + \frac{1}{2}\int J_1\Delta_{12}J_2 + \left(\frac{1}{2}\right)^2\frac{1}{2!}\int (J_1\Delta_{12}J_2)(J_3\Delta_{34}J_4) + \left(\frac{1}{2}\right)^3\frac{1}{3!}\int (J_1\Delta_{12}J_2)(J_3\Delta_{34}J_4)(J_5\Delta_{56}J_6) + \cdots$$

where

$$J_i = J_i\left(y_i\right), \qquad \Delta_{ij} = \triangle(y_i, y_j), \qquad \int = \int\prod_i d^4y_i$$

Now we look at z_1, the first order terms in λ,

$$z_1 = -\frac{1}{4!}\frac{1}{Z_0[J]}\{\int d^4x\left[\frac{\delta}{\delta J(x)}\right]^4\}Z_0\left[J\right]$$

Suppose we want to calculate the 2 point function

$$\left.\frac{\delta^2 Z\left[J\right]}{\delta J\left(x_1\right)\delta J\left(x_2\right)}\right|_{j=0}$$

which involves 2 differentiation with respect to J. Since there are 4 differentiation in z_1, we need to get terms in $Z_0\left[J\right]$ which are 6th order in J

$$Z_0^{(6)}\left[J\right] = \left(\frac{1}{2}\right)^3\frac{1}{3!}\int (J_1\Delta_{12}J_2)(J_3\Delta_{34}J_4)(J_5\Delta_{56}J_6)$$

We denote this symbolically as

$$Z_0^{(6)} = \left(\frac{1}{2}\right)^3\frac{1}{3!}\left(\begin{array}{c}(J_1 - - - J_2)\\(J_3 - - - J_4)\\(J_5 - - - J_6)\end{array}\right)$$

Then one differentiation gives

$$\frac{\delta}{\delta J(x)} Z_0^{(6)}[J] = \left(\frac{1}{2}\right)^3 \frac{1}{3!} \int 6(\Delta_{x2} J_2)(J_3 \Delta_{34} J_4)(J_5 \Delta_{56} J_6)$$

$$\frac{\delta}{\delta J} Z_0^{(6)} = \left(\frac{1}{2}\right)^3 \frac{1}{3!} \left[6 \left(\begin{array}{c} (x - - - J_2) \\ (J_3 - - - J_4) \\ (J_5 - - - J_6) \end{array} \right) \right]$$

Simmilarly, the second derivative is

$$\left(\frac{\delta}{\delta J(x)}\right)^2 Z_0^{(6)}[J] = \left(\frac{1}{2}\right)^3 \frac{1}{3!} \int 6[(\Delta_{xx})(J_3 \Delta_{34} J_4)(J_5 \Delta_{56} J_6) + 4(\Delta_{x2} J_2)(\Delta_{x4} J_4)(J_5 \Delta_{56} J_6)]$$

and symbolically

$$\frac{\delta^2}{\delta J^2} Z_0^{(6)} = \left(\frac{1}{2}\right)^3 \frac{1}{3!} 6 \left[\left(\begin{array}{c} (x - - - x) \\ (J_3 - - - J_4) \\ (J_5 - - - J_6) \end{array} \right) + 4 \left(\begin{array}{c} (x - - - J_2) \\ (x - - - J_4) \\ (J_5 - - - J_6) \end{array} \right) \right]$$

The third derivative is

$$\left(\frac{\delta}{\delta J(x)}\right)^3 Z_0^{(6)}[J] = \left(\frac{1}{2}\right)^3 \int [4(\Delta_{xx})(\Delta_{x4} J_4)(J_5 \Delta_{56} J_6) + 4 \times 2(\Delta_{xx})(\Delta_{x4} J_4)(J_5 \Delta_{56} J_6) + 4 \times 2(\Delta_{x2} J_2)(\Delta_{x4} J_4)(\Delta_{x6}$$

$$= \left(\frac{1}{2}\right)^3 \int [12(\Delta_{xx})(\Delta_{x4} J_4)(J_5 \Delta_{56} J_6) + 8(\Delta_{x2} J_2)(\Delta_{x4} J_4)(\Delta_{x6} J_6)]$$

and symbolically

$$\frac{\delta^3}{\delta J^3} Z_0^{(6)} = \left(\frac{1}{2}\right)^3 \left[12 \begin{pmatrix} (x - - - x) \\ (x - - - J_4) \\ (J_5 - - - J_6) \end{pmatrix} + 4 \begin{pmatrix} (x - - - J_2) \\ (x - - - J_4) \\ (x - - - J_6) \end{pmatrix} \right]$$

The 4th derivative is

$$\left(\frac{\delta}{\delta J(x)}\right)^4 Z_0^{(6)}[J] = \left(\frac{1}{2}\right)^3 \int [12(\Delta_{xx})(\Delta_{xx})(J_5 \Delta_{56} J_6) + 12 \times 2(\Delta_{xx})(\Delta_{x4} J_4)(\Delta_{x6} J_6) + 8 \times 3(\Delta_{xx})(\Delta_{x4} J_4)(\Delta_{x6} J_6)]$$
$$= \int [\frac{3}{2}(\Delta_{xx})^2 (J_5 \Delta_{56} J_6) + 6(\Delta_{xx})(\Delta_{x4} J_4)(\Delta_{x6} J_6)]$$

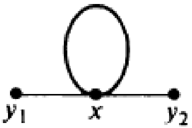
and symbolically

$$\frac{\delta^4}{\delta J^4} Z_0^{(6)} = \left[\frac{3}{2} \begin{pmatrix} (x - - - x) \\ (x - - - x) \\ (J_5 - - - J_6) \end{pmatrix} + 6 \begin{pmatrix} (x - - - x) \\ (x - - - J_4) \\ (x - - - J_6) \end{pmatrix} \right]$$

We see that the first term will give a disconnected graph to the 2-point function. So the connected part of the 2-point function to this order is

$$\left.\frac{\delta^2 Z[J]}{\delta J(z_1)\delta J(z_2)}\right|_{j=0} = \frac{\lambda}{4!} \frac{\delta^2}{\delta J(z_1)\delta J(z_2)} \{\int d^4x \left[\frac{\delta}{\delta J(x)}\right]^4\} Z_0[J] \Bigg|_{J=0} = \frac{\lambda}{2} \int d^4x \Delta(x,x)\Delta(z_1,x)\Delta(z_2,x)$$

This corresponds to the following graph



The first term corresponds to the disconnected graph

$$\langle 0 \,|\, T\left(\phi\left(y_1\right)\phi\left(y_2\right)\right)|\, 0\rangle|_{disc} = \frac{\delta^2}{\delta J\left(y_1\right)\delta J\left(y_2\right)}\lambda\int\left[\frac{3}{2}\left(\Delta_{xx}\right)^2\left(J_5\Delta_{56}J_6\right)\right|_{j=0} = 3\lambda(\Delta(x,x)^2\Delta\left(y_1,y_2\right)$$

We will now demonstrate that this disconnected graph will be canceled in the combination $W = -i\ln Z$. First we write

$$\frac{\delta W\left(J\right)}{\delta J_1} = -i\frac{1}{Z}\frac{\delta Z}{\delta J_1}, \qquad \frac{\delta^2 W\left(J\right)}{\delta J_1\delta J_2} = -i\left(-\frac{1}{Z^2}\frac{\delta Z}{\delta J_1}\frac{\delta Z}{\delta J_2} + \frac{1}{Z}\frac{\delta^2 Z}{\delta J_1\delta J_2}\right)$$

If we set $J = 0$, we find

$$\left.\frac{\delta^2 W\left(J\right)}{\delta J_1\delta J_2}\right|_{J=0} = -i\,\frac{1}{Z}\left.\frac{\delta^2 Z}{\delta J_1\delta J_2}\right|_{J=0}$$

The second factor is what we have just computed. For the factor Z, we can write

$$Z\left[J\right] = \left[\exp\int d^4x\frac{\lambda}{4!}\left(\frac{\delta}{\delta J\left(x\right)}\right)^4\right]Z_0\left[J\right] = Z_0\left[J\right]\left\{1 + \lambda z_1\left[J\right] + \ldots\right\}$$

where

$$z_1 = \frac{1}{4!}\frac{1}{Z_0[J]}\{\int d^4x \left[\frac{\delta}{\delta J(x)}\right]^4\} Z_0[J]$$

$$Z_0[J] = 1 + \frac{1}{2}\int J_1\Delta_{12}J_2 + \left(\frac{1}{2}\right)^2 \frac{1}{2!}\int (J_1\Delta_{12}J_2)(J_3\Delta_{34}J_4) + \left(\frac{1}{2}\right)^3 \frac{1}{3!}\int (J_1\Delta_{12}J_2)(J_3\Delta_{34}J_4)(J_5\Delta_{56}J_6) + \cdots$$

Clearly only J^4 terms in Z_0 will contribute to z_1

$$\frac{\delta}{\delta J(x)} Z_0^{(4)}[J] = \left(\frac{1}{2}\right)^2 \frac{1}{2!}\int 4(\Delta_{x2}J_2)(J_3\Delta_{34}J_4),$$

$$\frac{\delta^2}{\delta J(x)^2} Z_0^{(4)}[J] = \frac{1}{2!}\int [(\Delta_{xx})(J_3\Delta_{34}J_4) + 2(\Delta_{x2}J_2)(\Delta_{x4}J_4)]$$

$$\frac{\delta^3}{\delta J(x)^3} Z_0^{(4)}[J] = \frac{1}{2!}\int [2(\Delta_{xx})(\Delta_{x4}J_4) + 2\times 2(\Delta_{xx})(\Delta_{x4}J_4)]$$

$$\frac{\delta^4}{\delta J(x)^4} Z_0^{(4)}[J] = \frac{1}{2!}\int [2(\Delta_{xx})(\Delta_{xx}) + 2\times 2(\Delta_{xx})(\Delta_{xx})]$$
$$= 3(\Delta_{xx})^2$$

$$\{\int d^4x \left[\frac{\delta}{\delta J(x)}\right]^4\} Z_0^{(4)}[J]_{J=0} = \int d^4x 3\Delta^2(x,x)$$

Then we get

$$1 + \lambda z_1 = 1 + \frac{\lambda}{8}\int d^4x \Delta^2(x,x)$$

and

$$Z^{-1} = 1 - \frac{\lambda}{8} \int d^4 x \Delta^2 (x, x)$$

From

$$\left. \frac{\delta^2 Z}{\delta J_1 \delta J_2} \right|_{J=0} = \Delta (z_1, z_2) + \frac{\lambda}{2} \int d^4 x [\Delta(x,x)\Delta(z_1,x)\Delta(z_2,x) + \frac{1}{8} (\Delta(x,x)^2 \Delta (z_1, z_2)]$$

We get

$$\begin{aligned} \left. \frac{\delta^2 W(J)}{\delta J_1 \delta J_2} \right|_{J=0} &= -i \, \frac{1}{Z} \left. \frac{\delta^2 Z}{\delta J_1 \delta J_2} \right|_{J=0} = \Delta (z_1, z_2) + \frac{\lambda}{2} \int d^4 x [\Delta(x,x)\Delta(z_1,x)\Delta(z_2,x) + \frac{1}{8} (\Delta(x,x)^2 \Delta (z_1, z_2)] \\ &\times [1 - \frac{1}{8} \int d^4 x \Delta^2 (x, x)] \\ &= \Delta (z_1, z_2) + \frac{\lambda}{2} \int d^4 x [\Delta(x,x)\Delta(z_1,x)\Delta(z_2,x) \end{aligned}$$

Thus the disconnected diagrams cancell out and only the connected diagram remains. $Z_0 [J]$
For convenience we can work with the generating functional $Z_0 [J]$ and just throw away the disconnected graphs. To compute the 4-point function

$$\left. \frac{\delta^2 Z [J]}{\delta J (x_1) \delta J (x_2) \delta J (x_3) \delta J (x_4)} \right|_{j=0}$$

we need the 8th order terms in Z,

$$Z_0^{(8)} [J] = \left(\frac{1}{2} \right)^4 \frac{1}{4!} \int (J_1 \Delta_{12} J_2)(J_3 \Delta_{34} J_4)(J_5 \Delta_{56} J_6)(J_7 \Delta_{78} J_8)$$

Then

$$\frac{\delta}{\delta J(x)} Z_0^{(8)}[J] = \left(\frac{1}{2}\right)^4 \frac{1}{4!} \int 8(\Delta_{x2} J_2)(J_3 \Delta_{34} J_4)(J_5 \Delta_{56} J_6)(J_7 \Delta_{78} J_8)$$

$$\left(\frac{\delta}{\delta J(x)}\right)^2 Z_0^{(8)}[J] = \left(\frac{1}{2}\right)^4 \frac{1}{4!} \int 8[(\Delta_{xx})(J_3 \Delta_{34} J_4)(J_5 \Delta_{56} J_6)(J_7 \Delta_{78} J_8) + 6(\Delta_{x2} J_2)(\Delta_{x4} J_4)(J_5 \Delta_{56} J_6)(J_7 \Delta_{78} J_8)]$$

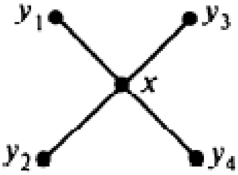
$$\begin{aligned}\left(\frac{\delta}{\delta J(x)}\right)^3 Z_0^{(8)}[J] &= \left(\frac{1}{2}\right)^4 \frac{1}{4!} \int 8[(\Delta_{xx})6(\Delta_{x4} J_4)(J_5 \Delta_{56} J_6)(J_7 \Delta_{78} J_8) + 6 \times 2(\Delta_{xx})(\Delta_{x4} J_4)(J_5 \Delta_{56} J_6)(J_7 \Delta_{78} J_8) \\ &+ 6 \times 4(\Delta_{x2} J_2)(\Delta_{x4} J_4)(\Delta_{x6} J_6)(J_7 \Delta_{78} J_8)] \\ &= \left(\frac{1}{2}\right)^4 \frac{1}{4!} \int 8[18(\Delta_{xx})(\Delta_{x4} J_4)(J_5 \Delta_{56} J_6)(J_7 \Delta_{78} J_8) + 24(\Delta_{x2} J_2)(\Delta_{x4} J_4)(\Delta_{x6} J_6)(J_7 \Delta_{78} J_8)]\end{aligned}$$

$$\begin{aligned}\left(\frac{\delta}{\delta J(x)}\right)^4 Z_0^{(8)}[J] &= \left(\frac{1}{2}\right)^4 \frac{1}{4!} \int 8[18(\Delta_{xx})(\Delta_{xx})(J_5 \Delta_{56} J_6)(J_7 \Delta_{78} J_8) + 18 \times 4(\Delta_{xx})(\Delta_{x4} J_4)(\Delta_{x6} J_6)(J_7 \Delta_{78} J_8) \\ &+ 24 \times 3(\Delta_{xx})(\Delta_{x4} J_4)(\Delta_{x6} J_6)(J_7 \Delta_{78} J_8) + 24 \times 2(\Delta_{x2} J_2)(\Delta_{x4} J_4)(\Delta_{x6} J_6)(\Delta_{x8} J_8)] \\ &= \left(\frac{1}{2}\right)^4 \frac{1}{4!} \int [96(\Delta_{xx})^2(J_5 \Delta_{56} J_6)(J_7 \Delta_{78} J_8) + 96 \times 8(\Delta_{xx})(\Delta_{x4} J_4)(\Delta_{x6} J_6)(J_7 \Delta_{78} J_8) \\ &+ 48 \times 8(\Delta_{x2} J_2)(\Delta_{x4} J_4)(\Delta_{x6} J_6)(\Delta_{x8} J_8)]\end{aligned}$$

It is not hard to see that the first 2 terms which contain Δ_{xx} correspond to disconnected graphs. Only the last term gives the connected 4-point function

$$\left.\frac{\delta^2 Z[J]}{\delta J(y_1)\,\delta J(y_2)\,\delta J(y_3)\,\delta J(y_4)}\right|_{j=0} = \frac{\lambda}{4!} \left\{\int d^4x \left[\frac{\delta}{\delta J(x)}\right]^4\right\} Z_0^{(8)}[J]\Bigg|_{J=0} = \lambda(\Delta_{xy_1})(\Delta_{xy_2})(\Delta_{xy_3})(\Delta_{xy_4})$$

This is shown graphicall below,



Here we see that in the path integral formalism we get the pertubation expansion which is exactly the same as the canonical quantization using Wicks theorem and we get the same Feynamn rules. We note that the generation function $Z[J]$ contains both the connected and disconnected diagrams. But only the connected graphs are of physical interest.

The connected Green's function is

$$G^{(n)}(x_1, x_2, ...x_n) = \frac{\delta^n \ln W[J]}{\delta J(x_1)\delta J(x_2)...\delta J(x_n)}|_{J=0}$$

Grassmann algebra

For fermion fields, we need to use anti-commuting c-number functions.This can be realized as elements of Grassmann algebra.

In an n-dimensional Grassmann algebra,the n generators θ_1, θ_2, $\theta_3, ...,$ θ_n satisfy the anti-commutation relations,

$$\{\theta_i, \theta_j\} = 0 \qquad i, j = 1, 2, ..., n$$

and every element can be expanded in a finite series,

$$P(\theta) = P_0 + P^{(1)}_{i_1}\theta_{i_1} + P^{(2)}_{i_1 i_2}\theta_{i_1}\theta_{i_2} + ... + P_{i_1 ... i_n}\theta_{i_1} ...\theta_{i_n}$$

Simplest case:n=1

$$\{\theta, \theta\} = 0 \qquad \text{or} \qquad \theta^2 = 0 \qquad P(\theta) = P_0 + \theta P_1$$

We can define the "differentiation" and "integration" as follows,

$$\frac{d}{d\theta}\theta = \theta\frac{\overleftarrow{d}}{d\theta} = 1 \quad \Longrightarrow \frac{d}{d\theta}P(\theta) = P_1$$

Integration is defined in such a way that it is invariant under translation,

$$\int d\theta P(\theta) = \int d\theta P(\theta + \alpha)$$

α is another Grassmann variable. This implies

$$\int d\theta = 0$$

We can normalize the integral

$$\int d\theta\theta = 1$$

Then

$$\int d\theta P(\theta) = P_1 = \frac{d}{d\theta} P(\theta)$$

Consider a change of variable

$$\theta \longrightarrow \widetilde{\theta} = a + b\theta$$

Since

$$\int d\widetilde{\theta} P\left(\widetilde{\theta}\right) = \frac{d}{d\widetilde{\theta}} P\left(\widetilde{\theta}\right) = P_1$$

$$\int d\theta P\left(\widetilde{\theta}\right) = \int d\theta \left[P_0 + \widetilde{\theta} P_1\right] = \int d\theta \left[P_0 + (a + b\theta) P_1\right] = bP_1$$

we get

$$\int d\widetilde{\theta} P\left(\widetilde{\theta}\right) = \int d\theta \left(\frac{d\widetilde{\theta}}{d\theta}\right)^{-1} P\left(\widetilde{\theta}(\theta)\right)$$

The "Jacobian" is the inverse of that for c-number integration.

Generalize to n-dimensional Grassmann algebra,

$$\frac{d}{d\theta_i}\left(\theta_1,\theta_2,\theta_3,...,\theta_n\right)=\delta_{i_1}\theta_2...\theta_n-\delta_{i_2}\theta_1\theta_3...\theta_n+...+\left(-1\right)^{n-1}\delta_{in}\theta_1\theta_2...\theta_{n-1}$$

$$\left\{d\theta_i,d\theta_j\right\}=0$$

$$\int d\theta_i=0 \qquad \int d\theta_i\theta_j=\delta_{ij}$$

For a change of variables of the form

$$\widetilde{\theta}_i=b_{ij}\theta_j$$

we have

$$\int d\widetilde{\theta}_n d\widetilde{\theta}_{n-1}...d\widetilde{\theta}_1 P\left(\widetilde{\theta}\right)=\int d\theta_n...d\theta_1\left[det\frac{d\widetilde{\theta}}{d\theta}\right]^{-1}P\left(\widetilde{\theta}\left(\theta\right)\right)$$

Proof:

$$\widetilde{\theta}_1\widetilde{\theta}_2...\widetilde{\theta}_n=b_{1i_1}b_{2i_2}...b_{ni_n}\theta_{i_1}...\theta_{i_n}$$

RHS is non-zero only if $i_1,i_2...,i_n$ are all different and we can write

$$\begin{aligned}\widetilde{\theta}_1\widetilde{\theta}_2...\widetilde{\theta}_n &= b_{1i_1}b_{2i_2}...b_{ni_n}\epsilon_{i_1,i_2...,i_n}\theta_{i_1}...\theta_{i_n}\\ &= \left(\det b\right)\theta_1\theta_2\theta_3...\theta_n\end{aligned}$$

From the normalization condition,

$$1=\int d\widetilde{\theta}_n d\widetilde{\theta}_{n-1}...d\widetilde{\theta}_1\left(\widetilde{\theta}_1\widetilde{\theta}_2...\widetilde{\theta}_n\right)=\left(\det b\right)\int d\widetilde{\theta}_n d\widetilde{\theta}_{n-1}...d\widetilde{\theta}_1\left(\theta_1\theta_2\theta_3...\theta_n\right)$$

we see that

$$d\widetilde{\theta}_n d\widetilde{\theta}_{n-1}...d\widetilde{\theta}_1=\left(\det b\right)^{-1}d\theta_1...d\theta_n$$

In field theory, we need Gaussian integral of the form,

$$G(A) \equiv \int d\theta_n ... d\theta_1 \exp\left(\frac{1}{2}(\theta, A\theta)\right) \qquad \text{where } (\theta, A\theta) = \theta_i A_{ij} \theta_j$$

First consider n=2

$$A = \begin{pmatrix} 0 & A_{12} \\ -A_{12} & 0 \end{pmatrix}$$

Then

$$G(A) = \int d\theta_2 d\theta_1 \exp(\theta_1 \theta_2 A_{12}) \simeq \int d\theta_2 d\theta_1 (1 + \theta_1 \theta_2 A_{12}) = A_{12} = \sqrt{\det A}$$

For the general $n = even$, we first bring the matrix A into the standard form by a unitary transformation,

$$UAU^\dagger = A_s$$

$$A_s = \begin{bmatrix} a\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \\ & & \ddots \end{bmatrix}$$

This can be seen as follows. Since iA is Hermitian, it cab diagonalized by a unitary transfomation,

$$V(iA)V^\dagger = A_d$$

where A_d is real and diagonal. The diagonal elements are solutions to the secular equation,

$$\det |iA - \lambda I| = 0$$

Since $A = -A^T$, we have

$$0 = \det |iA - \lambda I|^T = \det |-iA - \lambda I|$$

This means that if λ is a solution, -λ is also a solution and A_d is of the form,

$$A_d = \begin{pmatrix} a & & & & \\ & -a & & & \\ & & b & & \\ & & & -b & \\ & & & & \ddots \end{pmatrix}$$

To put this matrix into the standard we use the unitary matrix

$$S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$$

which has the property

$$S_2 \left(-i\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S_2^\dagger = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Thus we get

$$S \left(-iA_d\right) S^\dagger = A_s, \quad \text{where} \quad S = \begin{pmatrix} S_2 & & \\ & S_2 & \\ & & \ddots \end{pmatrix}$$

For arbitrary n, we get

$$G\left(A\right)=\int d\theta_{n}...d\theta_{1}\exp\left(\frac{1}{2}\left(\theta,A\theta\right)\right)=\sqrt{\det A}\qquad \text{n even}$$

and for "complex" Grassmann variables

$$\int d\theta_{n}d\bar{\theta}_{n}d\theta_{n-1}d\bar{\theta}_{n-1}...d\theta_{1}d\bar{\theta}_{1}\exp\left(\bar{\theta},A\theta\right)=\det A$$

For the Fermion fields, the generating functional is of the form,

$$W\left[\eta,\overline{\eta}\right]=\int\left[d\psi\left(x\right)\right]\left[d\overline{\psi}\left(x\right)\right]\exp\{i\int d^{4}x\left[\mathcal{L}\left(\psi,\overline{\psi}\right)+\overline{\psi}\eta+\overline{\eta}\psi\right]\}$$

If $\mathcal{L}$ depends on $\psi,\overline{\psi}$ quadratically

$$\mathcal{L}=\left(\overline{\psi},A\psi\right)$$

then we have

$$W=\int\left[d\psi\left(x\right)\right]\left[d\overline{\psi}\left(x\right)\right]\exp\{\int d^{4}x\overline{\psi}A\psi\}=\det A$$