Symmetry and Conservation Laws

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Fundamental Interactions

1. Strong Interaction–Quantum Chromodynamics (QCD)
   Local symmetry(Gauge Theory) based on $SU(3)$ color symmetry
2. Electromagnetic Interaction–Quantum Electrodynamics (QED)
   Local symmetry based on $U(1)$ symmetry
3. Weak interaction–
   Combine with QED to form Electroweak Theory
   Local symmetry based on $SU(2) \times U(1)$ symmetry
4. Gravity–Einstein's General Relativity
   Local symmetry–geneal coordinate transformtion

History

- Non-Abelian Gauge Theory– Yang-Mills 1954
- Spontaneous Symmetry Breaking (SSB)— Nambu, Goldstone, Salam, Weinberg, ~1960's
- SSB + Gauge theory– Higgs, Englert and Brout, Guralnik, Hagen, and Kibble, Anderson~1964
- Renormalization of Yang-Mills theory– Fadeev and Popov, t' Hooft 1971
- Standard Model–Electroweak Model– Weinberg, Salam, 1967

Symmetries and Conservation Laws

Symmetry
Symmetries play important roles in high energy physics. Symmetry $\Longrightarrow$ conservation law
Conservation Laws–all come from experiments directly or indirectly

1. Exact
   1. Energy Conservation–time translation
   2. Momentum Conservation–spatial translation
   3. Electric Charge
   4. Baryon Number
2. Approximate–Valid only in some approximations
   1. Parity
   2. Charge Conjugation
   3. Lepton Number
   4. Isospin

Example 1: Energy Conservation
For simple case, Newton's law gives

$$m\frac{d^2\vec{x}}{dt^2}=\vec{f}\left(\vec{x},t\right)$$

If $\vec{f}\left(\vec{x},t\right)$ is independent of t and $\vec{f}\left(\vec{x},t\right)=-\vec{\nabla}V\left(\vec{x}\right)$, then

$$m\frac{d^2\vec{x}}{dt^2}\cdot\frac{d\vec{x}}{dt}=-\vec{\nabla}V\left(\vec{x},t\right)\cdot\frac{d\vec{x}}{dt}\qquad\Longrightarrow\quad\frac{d}{dt}\left[\frac{1}{2}m\left(\frac{d\vec{x}}{dt}\right)^2+V\right]=0$$

Example 2 : rotational invariance
In Newton's equation with potential

$$\frac{d\vec{p}}{dt}=-\vec{\nabla}V\left(\vec{x}\right)$$

if $V\left(\vec{x}\right)$ is rotational invariant, i.e. $V\left(\vec{x}\right)=V\left(r\right)$, then

$$\vec{\nabla}V\left(\vec{x}\right)=\vec{\nabla}V\left(r\right)=\frac{dV}{dr}\hat{r}$$

and

$$\vec{r}\times\frac{d\vec{p}}{dt}=\frac{d}{dt}\left(\vec{r}\times\vec{p}\right)=-\frac{dV}{dr}\vec{r}\times\hat{r}=0$$

This gives angular momentum conservation.
Example 3: Momentum conservation

Consider 2 particles interact with each other through potential energy $V\left(\vec{x}_1 - \vec{x}_2\right)$ so that the Lagrangian is of the form,

$$L = \frac{m_1}{2}\left(\frac{d\vec{x}_1}{dt}\right)^2 + \frac{m_2}{2}\left(\frac{d\vec{x}_2}{dt}\right)^2 - V\left(\vec{x}_1 - \vec{x}_2\right)$$

This clearly invariant under the spatial translation,

$$\vec{x}_1 \longrightarrow \vec{x}_1' = \vec{x}_1 + \vec{a}, \qquad \vec{x}_2 \longrightarrow \vec{x}_2' = \vec{x}_2 + \vec{a},$$

The equations of motion are given by

$$\frac{d\vec{p}_1}{dt} = -\vec{\nabla}_1 V\left(\vec{x}_1 - \vec{x}_2\right)$$

$$\frac{d\vec{p}_2}{dt} = -\vec{\nabla}_2 V\left(\vec{x}_1 - \vec{x}_2\right)$$

Thus we have the momentum conservation,

$$\frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right) = 0$$

Remark: In the Maxwell equations we have current conservation,

$$\partial_\mu j^\mu = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

Define the total charge by

$$Q = \int_V d^3 x \rho$$

Then

$$\frac{dQ}{dt} = \int_V d^3 x \frac{\partial \rho}{\partial t} = -\int_V d^3 x \vec{\nabla} \cdot \vec{j} = -\int d\vec{S} \cdot \vec{j} = 0$$

where we have used Gauss theorem and assume $\vec{j} = 0$ on the surface.

Internal Symmetry-symmetry transformation in abstract space

Example: isospin symmetry

Motivation: nuclear force seems to be the same for neutron and proton

symmetry transfomation:

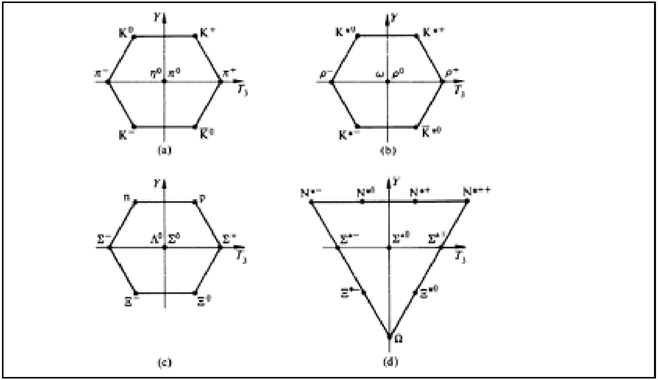
$$\begin{pmatrix} n(x) \\ p(x) \end{pmatrix} \rightarrow U \begin{pmatrix} n(x) \\ p(x) \end{pmatrix}, \qquad 2 \times 2 \quad \text{unitay matrix indep of } x^{\mu}$$

Consequence: $m_p = m_n$ degenerate states

Similarly, $\{\pi^-, \pi^0, \pi^+\}$, $\quad I = 1 \qquad$ triplet,

$\{K^0, K^+\}$, $\qquad I = 1/2 \qquad$ doublet

Eight-fold way : Gell-Mann, Neeman
Group togather mesons or baryons with same spin and parity,



These are the same as irreducible representations of $SU(3)$ group. The spectra of $SU(3)$ symmetry is not as good as isospin of $SU(2)$. Nevertheless, it is still useful to classify hadrons in terms of $SU(3)$ symmetry. This is known as the **eight-fold way**.

Quark Model

One peculiar feature of eight fold way is that octet and decuplet are not the fundamental representation of $SU(3)$ group. In 1964, Gell-mann and Zweig independently proposed the quark model: all hadrons are built out of spin $\frac{1}{2}$ quarks which transform as the fundamental representation of $SU(3)$,

$$q_i = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

with the quantum numbers

	Q	T	T_3	Y	S	B
u	$2/3$	$1/2$	$+1/2$	$1/3$	0	$1/3$
d	$-1/3$	$1/2$	$-1/2$	$1/3$	0	$1/3$
s	$-1/3$	0	0	$-2/3$	-1	$1/3$

In this scheme, mesons are $q\bar{q}$ bound states. For examples,

$$\pi^+ \sim \bar{d}u \quad \pi^0 \sim \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d). \quad \pi^- \sim \bar{u}d$$

$$K^+ \sim \bar{s}u \quad K^0 \sim \bar{s}d \;,\; K^- \sim \bar{u}s. \quad \eta^0 \sim \frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s)$$

and baryons are qqq bound states,

$$\begin{aligned} p &\sim uud \;,\; n \sim ddu \\ \Sigma^+ &\sim suu \;,\; \Sigma^0 \sim s\left(\frac{ud+du}{\sqrt{2}}\right) \;,\; \Sigma^- \sim sdd \\ \Xi^0 &\sim ssu \;,\; \Xi^- \sim ssd \;,\; \Lambda^0 \sim \frac{s(ud-du)}{\sqrt{2}}. \end{aligned}$$

Quantum numbers of the hadrons are all carried by the quarks. But we do not know the dynamics which bound the quarks into hadrons. Since quarks are the fundamental constituent of hadrons it is important to find these particles. But over the years none have been found.

Paradoxes of simple quark model

1. Quarks have fractional charges while all observed particles have integer charges $\Longrightarrow$ one of the quarks is stable. None has been found.
2. Hadrons are exclusively made out $q\bar{q}, qqq$ bound states. In other word, $qq, qqqq$ states are absent.
3. The quark content of the baryon N^{*++} is uuu. If the spin state is $\left|\frac{3}{2}, \frac{3}{2}\right\rangle$ then all quarks are in spin-up state~ $\alpha_1\alpha_2\alpha_3$ is totally symmetric. If we assume that the ground state has $l = 0$, then spatial wave function is also symmetric. This will leads to violation of Pauli exclusion principle.

Color degree of freedom

One way out of these problems, is to introduce color degrees of freedom for each quark and postulates that only color singlets are physical observables. 3 colors are needed to get antisymmetric wave function for N^{*++} and remains a color singlet state.

$$u_\alpha = (u_1, u_2, u_3) \quad , \quad d_\alpha = (d_1, d_2, d_3) \cdots$$

All hadrons form singlets under $SU(3)_{color}$ symmetry, e.g.

$$N^{*++} \sim u_\alpha(x_1)\alpha_\beta(x_2)u_\gamma(x_3)\varepsilon^{\alpha\beta\gamma}$$

Futhermore, color singlets can not be formed from the combination qq, $qqqq$ and they are absent from the observed specrum. Needless to say that a single quark is not observable.

Baryon number

Why proton is stable? $p \to e^+ + \gamma$ does not violate any physical laws
Postulate Baryon number conservation: $B(p) = 1$, $B(e^+) = 0$, $B(\gamma) = 0$,
In the universe at large, only baryons and no anti-baryons
At beginning, maybe $B = 0$ for the universe as whole, because

$$\gamma + \gamma \rightleftarrows p + \overline{p}$$

To get $B \neq 0$ now, we need baryon number non-sonservation (Sakharov)
In Grand Unified Theory, it is possible to have

$$p \to \pi^0 + e^+$$

Many experiments (IMB, Sudane, Kamiokonde...) search for this decay with null result,

$$\tau \left(p \to \pi^0 + e^+ \right) > 10^{31} \quad \text{years}$$

Symmetry and Noether's Theorem

Particle mechanics

First illustrate Noether's theorem in classical mechnics. The action is given by

$$S = \int L(q_i, \dot{q}_i)\, dt$$

Suppose S is invariant under some continuous symmetry transformation,

$$q_i \longrightarrow q_i' = f_{ij}(\alpha) q_j$$

where $f_{ij}(\alpha)'s$ are some functions of a parameter α,with $f_{ij}(0) = \delta_{ij}$. Consider infinitesmal transformation,

$$\alpha \ll 1$$

then,

$$q_i \longrightarrow q_i' \simeq q_i + \alpha f_{ij}'(0) q_j = q_i + \delta q_i \qquad \text{with} \quad \delta q_i = \alpha f_{ij}'(0) q_j$$

The change of S is

$$\delta S = \int [\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i]\, dt \quad \text{where} \quad \delta \dot{q}_i = \frac{d}{dt}(\delta q_i)$$

Using the equation of motion,

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})$$

we can write δS as

$$\delta S = \int [\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i)]\, dt = \int [\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}\delta q_i)]\, dt$$

Thus $\delta S = 0$ will yield

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) = 0 \quad \text{or} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\alpha f'_{ij}(0) q_j\right) = 0$$

This can be written as

$$\frac{dA}{dt} = 0, \quad \text{with} \qquad A = \frac{\partial L}{\partial \dot{q}_i}\alpha f'_{ij}(0) q_j$$

then A is the conserved quantity.
Example: Rotational symmetry in 3-dimension
Write the action as

$$S = \int L(\vec{x}, \dot{\vec{x}})\, dt = \int L(x_i, \dot{x}_i)\, dt$$

Suppose S is invariant under rotations in 3-dimension,

$$x_i \longrightarrow x'_i = R_{ij} x_j$$

where R is an orthogonal matrix, i.e.

$$RR^T = R^T R = 1 \quad \text{or} \quad R_{ij} R_{ik} = \delta_{jk}$$

For infinitesmal rotations, we write

$$R_{ij} = \delta_{ij} + \varepsilon_{ij}, \qquad |\varepsilon_{ij}| \ll 1$$

Orthogonality requires,

$$(\delta_{ij} + \varepsilon_{ij})(\delta_{ik} + \varepsilon_{ik}) = \delta_{jk} \Longrightarrow \varepsilon_{jk} + \varepsilon_{kj} = 0 \quad i, e, \quad \varepsilon_{jk} \quad \text{is} \quad \text{antisymmetric}$$

We can compute the conserved quantities as

$$J = \frac{\partial L}{\partial \dot{x}} \varepsilon_{ij} x_j = \varepsilon_{ij} p_i x_j$$

If we write $\varepsilon_{ij} = -\varepsilon_{ijk}\theta_k$, then

$$J = -\theta_k \varepsilon_{ijk} p_i x_j = -\theta_k J_k \qquad J_k = \varepsilon_{ijk} x_i p_j$$

Here J_k can be identified with k-th component of the usual angular momentum.

Field Theory

Start from the action,

$$S = \int L(\phi, \partial_\mu \phi)\, d^4x$$

Consider the symmetry transformation,

$$\phi(x) \to \phi'(x'),$$

where we have included the transformations which involve change of coordinates,

$$x^\mu \to x'^\mu$$

For infinitesmal transformation, we write

$$\delta\phi = \phi'(x') - \phi(x), \qquad \delta x'^\mu = x'^\mu - x^\mu$$

For the transformation involving changes of coordinates, we need to include the change in the volume element

$$d^4x' = Jd^4x \qquad \text{where} \quad J = \left| \frac{\partial(x_0', x_1', x_2', x_3')}{\partial(x_0, x_1, x_2, x_3)} \right|$$

is the Jacobian for the coordinate transformation. For infinitesmal transformation we can write,

$$J = \left|\frac{\partial x'^\mu}{\partial x^\nu}\right| \approx \left|g^\mu_\nu + \frac{\partial(\delta x^\mu)}{\partial x^\nu}\right| \approx 1 + \partial_\mu(\delta x^\mu)$$

where we have used the relation

$$det(1+\varepsilon) \approx 1 + Tr(\varepsilon) \qquad \text{for} \quad |\varepsilon| \ll 1$$

Then

$$d^4x' = d^4x(1 + \partial_\mu(\delta x^\mu))$$

The change in the action is then

$$\delta S = \int [\frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi) + L\partial_\mu(x^\mu)]\, dx^4$$

It is useful to define the change of ϕ for fixed x^μ,

$$\bar{\delta}\phi(x) = \phi'(x) - \phi(x) = \phi'(x) - \phi'(x') + \phi'(x') - \phi(x) = \phi'(x) - \phi'\ (x) + (\partial^\mu\phi')\delta x_\mu + \delta\phi$$

Note the operator $\bar{\delta}$ commutes with the derivative operator ∂_μ.

$$\delta\phi = \bar{\delta}\phi + (\partial_\mu\phi)\delta x^\mu$$

Similarly,

$$\delta(\partial_\mu\phi) = \bar{\delta}(\partial_\mu\phi) + \partial_\nu(\partial_\mu\phi)\delta x^\nu$$

Then

$$\delta S = \int [\frac{\partial L}{\partial \phi}(\bar{\delta}\phi + (\partial_\mu\phi)\delta x^\mu) + \frac{\partial L}{\partial(\partial_\mu \phi)}(\bar{\delta}(\partial_\mu\phi) + \partial_\nu(\partial_\mu\phi)\delta x^\nu) + L\partial_\mu(\delta x^\mu)]\, dx^4$$

Using Euler-Lagrange equation of motion

$$\frac{\partial L}{\partial \phi} = \partial^\mu(\frac{\partial L}{\partial(\partial_\mu \phi)})$$

we can write

$$\frac{\partial L}{\partial \phi}\bar{\delta}\phi + \frac{\partial L}{\partial(\partial_\mu \phi)})\bar{\delta}(\partial_\mu\phi) = \partial^\mu(\frac{\partial L}{\partial(\partial_\mu \phi)}\bar{\delta}\phi + \frac{\partial L}{\partial(\partial_\mu \phi)}\partial_\mu(\bar{\delta}\phi) = \partial^\mu[\frac{\partial L}{\partial(\partial_\mu \phi)}\bar{\delta}\phi]$$

where we have used

$$\partial_\mu(\bar{\delta}\phi) = \bar{\delta}(\partial_\mu \phi)$$

We can also combine other terms as

$$[\frac{\partial L}{\partial \phi}(\partial_\nu \phi) + \frac{\partial L}{\partial(\partial_\mu \phi)}\partial_\nu(\partial_\mu \phi)]\delta x^\nu + L\partial_\nu(\delta x^\nu) = (\partial_\nu L)\delta x^\nu + L\partial_\nu(\delta x^\nu) = \partial_\nu(L\delta x^\nu)$$

Then we get

$$\delta S = \int dx^4 \partial_\mu [\frac{\partial L}{\partial(\partial_\mu \phi)}\bar{\delta}\phi + L\delta x^\mu]$$

and if δS=0 under the symmetry ransformation of fields, then

$$\partial^\mu J_\mu = \partial^\mu [\frac{\partial L}{\partial(\partial_\mu \phi)}\bar{\delta}\phi + L\delta x^\mu] = 0 \qquad \text{current conservation}$$

Example: space-time translation
Here the coordinate transformation is,

$$x^\mu \to x'^\mu = x^\mu + a^\mu \Longrightarrow \phi'(x+a) = \phi(x)$$

then

$$\bar{\delta}\phi = \phi'^\mu \partial_\mu \phi$$

and the conservation laws take the form

$$\partial^\mu [\frac{\partial L}{\partial(\partial_\mu \phi)}(-a^\nu \partial_\nu \phi) + La^\mu] = -\partial^\mu (T_{\mu\nu} a^\nu) = 0$$

where

$$T_{\mu\nu} = \frac{\partial L}{\partial(\partial_\mu \phi)} \partial_\nu \phi - g_{\mu\nu} L$$

is the energy momentum tensor. In particular,

$$T_{0i} = \frac{\partial L}{\partial(\partial_0 \phi)} \partial_i \phi$$

and

$$P_i = \int dx^3 T_{0i}$$

is the total momentum of the fields. Also

$$T_{00} = \frac{\partial L}{\partial(\partial_0 \phi)} \partial_0 \phi - L$$

is the Hamiltonian density and

$$E = \int dx^3 T_{00}$$

is the total energy.