Global and Local symmetries

Ling-Fong Li

Group Theory

The most useful tool for studying symmetry is the group theory. We will give a simple discussion of the parts of the group theory which are commonly used in high energy physics.

Elements of group theory

A group G is a collection of elements (a, b, c$\cdots$) with a multiplication laws having the following properties;

1. Closure. If $a, b \in G$, $c = ab \in G$
2. Associative $a(bc) = (ab)c$
3. Identity $\exists e \in G \quad \ni \quad a = ea = ae \quad \forall a \in G$
4. Inverse For every $a \in G$, $\exists a^{-1} \quad \ni \quad aa^{-1} = e = a^{-1}a$

Examples of groups frequently used in physics are :

1. **Abelian group** —– group multiplication commutes, i.e. $ab = ba \quad \forall a, b \in G$
   e.g. cyclic group of order n, Z_n, consists of $a, a^2, a^3, \cdots, a^n = E$
2. **Orthogonal group** —— $n \times n$ orthogonal matrices, $RR^T = R^T R = 1$, $R : n \times n$ matrix
   e. g. the matrices representing rotations in 2-dimesions,

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

3. **Unitary group** ———– $n \times n$ unitary matrices,

We can built larger groups from smaller ones by direct product:

Direct product group —– Given any two groups , $G = \{g_1, g_2 \cdots\}$, $H = \{h_1, h_2 \cdots\}$ and if $g's$ commute with $h's$ we can define a direct product group by $G \times H = \{g_i h_j\}$ with multiplication law

$$(g_i h_j)(g_m h_n) = (g_i g_m)(h_j h_n)$$

Theory of Representation

Consider a group $G = \{g_1 \cdots g_n \cdots\}$. If for each group element g_i,there is an $n \times n$ matrix $D(g_i)$ such that it preserves the group multiplication, i.e.

$$D(g_1)D(g_2) = D(g_1 g_2) \qquad \forall \;\; g_1, g_2 \in G$$

then $D's$ forms a representation of the group G (n-dimensional representation). In other words, $g_i \longrightarrow D(g_i)$ is a homomorphism. If there exists a non-singular matric M such that all matrices in the representation can be transformed into block diagonal form,

$$MD(a)M^{-1} = \begin{pmatrix} D_1(a) & 0 & 0 \\ 0 & D_2(a) & 0 \\ & & \ddots \\ 0 & 0 & \end{pmatrix} \qquad \textit{for all } a \in G.$$

$D(a)$ is called reducible representation. If representation is not reducible, then it is irreducible representation (irrep)

Continuous group: groups parametrized by set of continuous parameters

Example: Rotations in 2-dimensions can be parametrized by $0 \le \theta < 2\pi$

SU(2) group

Set of 2×2 unitary matrices with determinant 1 is called $SU(2)$ group.
In general, $n \times n$ unitary matrix U can be written as

$$U = e^{iH} \qquad H : n \times n \ \text{ hermitian matrix}$$

From

$$\det U = e^{iTrH}$$

we get

$$TrH = 0 \qquad if \qquad detU = 1$$

Thus $n \times n$ unitary matrices U can be written in terms of $n \times n$ traceless Hermitian matrices.

Note that Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a complete set of 2×2 hermitian traceless matrices. We can use them to describe $SU(2)$ matrices.
Define $J_i = \frac{\sigma_i}{2}$. We can compute the commutators

$$[J_1, J_2] = iJ_3 \ , \ [J_2, J_3] = iJ_1 \ , \ [J_3, J_1] = iJ_2$$

This is the *Lie* algebra of $SU(2)$ symmetry. This is exactly the same as the commutation relation of angular momentum.

Irrep of $SU(2)$ algebra
Define

$$J^2 = J_1^2 + J_2^2 + J_2^3 \quad , \quad \text{with property} \quad [J^2, J_i] = 0 \quad , \quad i = 1, 2, 3$$

Also define

$$J_\pm \equiv J_1 \pm iJ_2 \quad \text{then} \quad J^2 = \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2 \quad \text{and} \quad [J_+, J_-] = 2J_3$$

For convenience, choose simultaneous eigenstates of J^2, J_3,

$$J^2|\lambda, m\rangle = \lambda|\lambda, m\rangle \quad , \quad \lambda_3|\lambda, m\rangle = m|\lambda, m\rangle$$

From

$$[J_+, J_3] = -J_+$$

we get

$$(J_+J_3 - J_3J_+)|\lambda, m\rangle = -J_+|\lambda, m\rangle$$

Or

$$J_3(J_+|\lambda, m\rangle) = (m+1)(J_+|\lambda, m\rangle)$$

Thus J_+ raises the eigenvalue from m to $m+1$ and is called *raising operator*. Similarly, J_- lowers m to $m-1$,

$$J_3(J_-|\lambda, m\rangle) = (m-1)(J_-|\lambda, m\rangle)$$

Since

$$J^2 \geq J_3^2 \quad , \quad \lambda - m^2 \geq 0$$

we see that m is bounded above and below. Let j be the largest value of m, then

$$J_+|\lambda, j\rangle = 0$$

Then

$$0 = J_- J_+ |\lambda, j\rangle = (J_3^2 - J_3^2 - J_3)|\lambda, j\rangle = (\lambda - j^2 - j)|\lambda, j\rangle$$

and

$$\lambda = j(j+1)$$

Similarly, let j' be the smallest value of m, then

$$J_- |\lambda, j'\rangle = 0 \qquad \lambda = j'(j'-1)$$

Combining these 2 relations, we get

$$j(j+1) = j'(j'-1) \quad \Rightarrow \quad j' = -j \;\; and \;\; j - j' = 2j = integer$$

We will use j, m to label the states. Assume the states are normalized,

$$\langle jm|jm'\rangle = \delta_{mm'}$$

Write

$$J_\pm |jm\rangle = C_\pm(jm)|j, m \pm 1\rangle$$

then

$$\langle jm|J_- J_+|jm\rangle = |C_+(j,m)|^2$$

$$LHS = \langle j, m|(J^2 - J_3^2 - J_3)|jm\rangle = j(j+1) - m^2 - m$$

This gives

$$C_+(j,m) = \sqrt{(j-m)(j+m+1)}$$

Similarly

$$C_-(j,m) = \sqrt{(j+m)(j-m+1)}$$

Summary: eigenstates $|jm\rangle$ have the properties

$$J_3|j,m\rangle = m|j,m\rangle \quad J_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}|jm\pm 1\rangle \quad , \quad J^2|j,m\rangle = j(j+1)jm\rangle$$

$J|j,m\rangle$, $m = -j, -j+1, \cdots, j$ are the basis for irreducible representation of SU(2) group. From these relations we can construct the representation matrices. We will illustrate these by following examples.

Example: $j = \frac{1}{2}$, $m = \pm\frac{1}{2}$

$$J_3 = |\frac{1}{2}, \pm\frac{1}{2}\langle = \pm\frac{1}{2}|\frac{1}{2}, \pm\frac{1}{2}\rangle$$

$$J_+|\frac{1}{2}, \frac{1}{2}\rangle = 0 \quad , \quad J_+|\frac{1}{2}, -\frac{1}{2} = |\frac{1}{2}, \frac{1}{2}\rangle \quad , \quad J_-|\frac{1}{2}, \frac{1}{2} = |\frac{1}{2}, -\frac{1}{2}\rangle \quad , \quad J_-|\frac{1}{2}, -\frac{1}{2}\rangle = 0$$

If we write

$$|\frac{1}{2}, \frac{1}{2}\rangle = \alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\frac{1}{2}, -\frac{1}{2}\rangle = \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Then we can represent $J's$ by matrices,

$$J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$J_1 = \frac{1}{2}(J_+ + J_-) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad J_2 = \frac{1}{2i}(J_+ - J_-) = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Within a factor of $\frac{1}{2}$, these are just Pauli matrices
Summary:

1. Among the generator only J_3 is diagonal, — SU(2) is a rank-1 group
2. Irreducible representation is labeled by j and the dimension is $2j+1$
3. Basis states $|j, m\rangle$ $m = j, j-1 \cdots (-j)$ representation matrices can be obtained from

$$J_3|j, m\rangle = m|j, m\rangle \qquad J_\pm|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$$

SU(2) and rotation group

The generators of $SU(2)$ group are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let $\overrightarrow{r} = (x, y, z)$ be arbitrary vector in R_3 (3 dimensional coordinate space). Define a 2×2 matrix h by

$$h = \vec{\sigma} \cdot \overrightarrow{r} = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}$$

h has the following properties

1. $h^{+} = h$
2. $Trh = 0$
3. $det \ h = -(x^2 + y^2 + z^2)$

Let U be a 2×2 unitary matrix with $detU = 1$. Consider the transformation

$$h \longrightarrow h' = UhU^{\dagger}$$

Then we have

1. $h'^{+} = h'$
2. $Trh' = 0$
3. $\det h' = \det h$

Properties (1)&(2) imply that h' can also be expanded in terms of Pauli matrices

$$h' = \vec{r}' \cdot \vec{\sigma} \ \overrightarrow{r} = (x', y', z')$$

$$\det h' = \det h \quad \Rightarrow \quad x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$$

Thus relation between $\overrightarrow{r}$ and $\overrightarrow{r}'$ is a rotation. This means that an arbitrary 2×2 unitary matrix U induces a rotation in R_3. This provides a connection between $SU(2)$ and $O(3)$ groups.

Rotation group & QM

Rotation in R_3 can be represented as linear transformations on

$$\vec{r} = (x, y, z) = (r_1, r_2, r_3) \quad , \quad r_i \rightarrow r_i' = R_{ij} X_j \qquad RR^T = 1 = R^T R$$

Consider an arbitary function of coordinates, $f(\vec{r}) = f(x, y, z)$. Under the rotation, the change in f

$$f(r_i) \rightarrow \ f(R_{ij} r_j) = f'(r_i)$$

If $f = f'$ we say f is invariant under rotation, eg $f(r_i) = f(r), r = \sqrt{x^2 + y^2 + z^2}$
In QM, we implement the rotation by

$$|\psi\rangle \rightarrow |\psi'\rangle = U|\psi\rangle, \quad O \rightarrow O' = UOU^\dagger$$

so that

$$\Rightarrow \langle\psi'|O'|\psi'\rangle = \langle\psi|O|\psi\rangle$$

If $O' = O$, we say O is invariant under rotation

$$\rightarrow \quad UO = OU \quad [O, U] = 0$$

In terms of infinitesimal generators, we have

$$U = e^{-i\theta\vec{n}\cdot\vec{J}}$$

This implies

$$[J_i, O] = 0, \ i = 1, 2, 3$$

For the case where O is the Hamiltonian H, this gives $[J_i, H] = 0$. Let $|\psi\rangle$ be an eigenstate of H with eigenvaule E,

$$H|\psi\rangle = E|\psi\rangle$$

then

$$(J_i H - H J_i)|\psi\rangle = 0 \quad \Rightarrow \quad H(J_i|\psi\rangle) = E(J_i|\psi\rangle)$$

$i.e$ $\quad |\psi\rangle$ & $J_i|\psi\rangle$ are degenerate. For example, let $|\psi\rangle = |j, m\rangle$ the eigenstates of angular momentum, then $J_\pm|j.m\rangle$ are also eigenstates if $|\psi\rangle$ is eigenstate of H. This means for a given j , the degeneracy is $(2j + 1)$.

Global symmetry in Field Theory

Example 1: Self interacting scalar fields
Consider Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2\right] - \frac{\mu^2}{2}\left(\phi_1^2 + \phi_2^2\right) - \frac{\lambda}{4}\left(\phi_1^2 + \phi_2^2\right)^2$$

this is invariant under rotation in (ϕ_1, ϕ_2) plane, $O(2)$ symmetry,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

θ is independent of x^μ and is called **global** transformation.
Physical consequences:

1. Mass degenercy
2. Relation between coupling constants

————————————————————————

Noether's currrent: for $\theta \ll 1$,

$$\delta\phi_1 = -\theta\phi_2, \qquad \delta\phi_2 = \theta\phi_1$$

and

$$J_\mu \sim \frac{\partial\mathcal{L}}{\partial\phi_i}\delta\phi_i = -\left[(\partial_\mu\phi_1)\,\phi_2 - (\partial_\mu\phi_2)\,\phi_1\right]$$

This current is conserved,

$$\partial_\mu J^\mu = 0$$

and conserved charge is

$$Q = \int d^3x J^0,$$

and

$$\frac{dQ}{dt} = \int d^3x \frac{\partial J^0}{\partial t} = -\int d^3x \vec{\nabla} \cdot \vec{J} = -\int d\vec{S} \cdot \vec{J} = 0$$

Another way is to write

$$\phi = \frac{1}{\sqrt{2}} \left(\phi_1 + i\phi_2 \right)$$

and

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial_\mu \phi - \mu^2 \phi^\dagger \phi - \lambda \left(\phi^\dagger \phi \right)^2$$

This is a phase transformation,

$$\phi \longrightarrow \phi' = e^{-i\theta} \phi$$

This is called the $U(1)$ symmetry. Charge conservation. is one such example. Approximate symmetries, e.g. lepton number, isospin, Baryon number,$\cdots$ are probably realized in the form of global symmtries.

Example 2 : Yukawa interaction–Scalar field interacting with fermion field
Lagrangian is of the form

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi + \frac{1}{2}\left(\partial_{\mu}\phi\right)^{2} - \frac{\mu^{2}}{2}\phi^{2} - \frac{\lambda}{4}\phi^{4} + g\bar{\psi}\gamma_{5}\psi\phi$$

This Lagrangian is invariant under the $U(1)$ transformation,

$$\psi \rightarrow \psi' = e^{i\alpha}\psi, \qquad \phi \rightarrow \phi' = \phi$$

Here the fermion number is conserved. Note that if there are two such fermions, ψ_1, ψ_2 with same transformation, then the Yukawa interaction will be

$$\mathcal{L}_{Y} = g_{1}\bar{\psi}_{1}\gamma_{5}\psi_{1}\phi + g_{2}\bar{\psi}_{2}\gamma_{5}\psi_{2}\phi$$

Thus we have two independent couplings g_1, g_2, one for each fermion.
Example 3 : Global non-abelian symmetry
Consider the case where ψ is a doublet and ϕ a singlet under $SU(2)$,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

and under $SU(2)$

$$\psi \rightarrow \psi' = \exp i\left(\frac{\vec{\tau}\cdot\vec{\alpha}}{2}\right)\psi, \qquad \phi \rightarrow \phi' = \phi$$

$\overrightarrow{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ are real parameters. The Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 + g\bar{\psi}\psi\phi$$

is $SU(2)$ invariant.
The Noether's currents are of the form,

$$\overrightarrow{J} = \bar{\psi}(\gamma^\mu \frac{\overrightarrow{\tau}}{2})\psi$$

and conserved charges are

$$Q^i = \int \psi^\dagger (\frac{\tau_i}{2})\psi$$

One can verify that

$$\left[Q^i, Q^j\right] = i\varepsilon^{ijk} Q^k$$

which is the $SU(2)$ algebra.

Local Symmetry

Local symmetry: transformation parameters, e.g. angle θ, depend on x^μ. This originates from electromagnetic theory.

Maxwell Equations:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}, \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0, \qquad \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} = \varepsilon_0\frac{\partial\vec{E}}{\partial t} + \vec{J}$$

Introduce $\phi, \vec{A}$ to solve those equations without source,

$$\vec{B} = \vec{\nabla}\times\vec{A}, \qquad \vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}$$

These are not unique because of **gauge** tranformation

$$\phi \longrightarrow \phi - \frac{\partial\alpha}{\partial t}, \qquad \vec{A} \longrightarrow \vec{A} + \vec{\nabla}\alpha$$

or

$$A_\mu \longrightarrow A_\mu - \partial_\mu\alpha$$

will give the same electromagnetic fields

In quantum mechanics, Schrodinger equation for charged particle,

$$\left[\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - e\vec{A}\right)^2 - e\phi\right]\psi = i\hbar\frac{\partial\psi}{\partial t}$$

This requires transformation of wave function,

$$\psi \longrightarrow \exp\left(i\frac{e}{\hbar}\alpha\left(x\right)\right)\psi$$

to get same physics.
Thus gauge transformation is connected to **symmetry** (local) transformation.

In field theory, gauge fields are needed to contruct covariant derivatives.

1) **Abelian symmetry**

Consider Lagrangian with global $U(1)$ symmetry,

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2$$

Suppose phase transformation depends on x^μ,

$$\phi \longrightarrow \phi' = e^{ig\alpha(x)} \phi$$

The derivative transforms as

$$\partial^\mu \phi \longrightarrow \partial^\mu \phi' = e^{i\alpha(x)} \left[\partial^\mu \phi + ig (\partial^\mu \alpha) \phi \right],$$

not a phase transformation.

Introduce gauge field A^μ, with transformation

$$A^\mu \longrightarrow A'^\mu = A^\mu - \partial^\mu \alpha$$

The combination

$$D^\mu \phi \equiv (\partial^\mu - igA^\mu) \phi, \qquad \text{covariant derivative}$$

will be transformed by a phase,

$$D^\mu \phi' = e^{ig\alpha(x)} (D^\mu \phi)$$

and the combination

$$D_\mu \phi^\dagger D^\mu \phi$$

is invarianat under local phase transformation.

Define anti-symmetric tensor for the gauge field

$$\left(D_\mu D_\nu - D_\nu D_\mu\right)\phi = gF_{\mu\nu}\phi, \qquad \text{with} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

We can use the property of the covariant derivative to show that

$$F'_{\mu\nu} = F_{\mu\nu}$$

Complete Lagragian is

$$\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(\phi)$$

where $V(\phi)$ does not depend on derivative of ϕ.

- mass term $A^\mu A_\mu$ is not gauge invariant $\Rightarrow$ massless particle$\Rightarrow$long range force
- coupling of gauge field to other field is universal

2) Non-Abelian symmetry-Yang Mills fields

1954: Yang-Mills generalized U(1) local symmetry to SU(2) local symmetry.

Consider an isospin doublet $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

Under SU(2) transformation

$$\psi(x) \to \psi'(x) = exp\{-\frac{i\vec{\tau}\cdot\vec{\theta}}{2}\}\psi(x)$$

where $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with

$$[\frac{\tau_i}{2}, \frac{\tau_j}{2}] = i\epsilon_{ijk}(\frac{\tau_k}{2})$$

Start from free Lagrangian

$$\mathcal{L}_0 = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi$$

which is invariant under global $SU(2)$ transformation where $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ are indep of x_μ.
For local symmetry transformation, write

$$\psi(x) \to \psi'(x) = U(\theta)\psi(x) \qquad U(\theta) = exp\{-\frac{i\vec{\tau}\cdot\theta(\vec{x})}{2}\}$$

Derivative term

$$\partial_\mu\psi(x) \to \partial_\mu\psi'(x) = U\partial_\mu\psi + (\partial_\mu U)\psi$$

is not invariant. Introduce gauge fields $\vec{A}_\mu$ to form the covariant derivative,

$$D_\mu \psi(x) \equiv (\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2})\psi$$

Require that

$$[D_\mu\psi]' = U[D_\mu\psi]$$

Or

$$(\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A}_\mu{}'}{2})(U\psi) = U(\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2})\psi$$

This gives the transformation of gauge field,

$$\boxed{\frac{\vec{\tau}\cdot\vec{A}_\mu{}'}{2} = U(\frac{\vec{\tau}\cdot\vec{A}_\mu}{2})U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}}$$

We use covariant derivatives to construct field tensor

$$D_\mu D_\nu\psi = (\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2})(\partial_\nu - ig\frac{\vec{\tau}\cdot\vec{A}_\nu}{2})\psi = \partial_\mu\partial_\nu\psi - ig(\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}\partial_\nu\psi + \frac{\vec{\tau}\cdot\vec{A}_\nu}{2}\partial_\mu\psi)$$

$$-ig\partial_\mu(\frac{\vec{\tau}\cdot\vec{A}_\nu}{2})\psi + (-ig)^2(\frac{\vec{\tau}\cdot\vec{A}_\mu}{2})(\frac{\vec{\tau}\cdot\vec{A}_\nu}{2})\psi$$

Antisymmetrize this to get the field tensor,

$$(D_\mu D_\nu - D_\nu D_\mu)\psi \equiv ig(\frac{\vec{\tau}\cdot\vec{F_{\mu\nu}}}{2})\psi$$

then

$$\frac{\vec{\tau}\cdot\vec{F_{\mu\nu}}}{2} = \frac{\vec{\tau}}{2}\cdot(\partial_\mu\vec{A_\nu} - \partial_\nu\vec{A_\mu}) - ig[\frac{\vec{\tau}\cdot\vec{A_\mu}}{2}, \frac{\vec{\tau}\cdot\vec{A_\nu}}{2}]$$

Or in terms of components,

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk}A^i_\mu A^k_\nu$$

The the term quadratic in A is new in Non-Abelian symmetry. Under the gauge transformation we have

$$\vec{\tau}\cdot\vec{F_\mu}\nu' = U(\vec{\tau}\cdot\vec{F_\mu}\nu)U^{-1}$$

Infinitesmal transformation $\theta(x) \ll 1$

$$A^{i/\mu} = A^{\mu} + \epsilon^{ijk}\theta^{j}A^{k}_{\mu} - \frac{1}{g}\partial_{\mu}\theta^{i}$$

$$F^{/i}_{\mu\nu} = F^{i}_{\mu\nu} + \epsilon^{ijk}\theta^{j}F^{k}_{\mu\nu}$$

Remarks

1. Again $A^a_\mu A^{a\mu}$ is not gauge invariant⇒gauge boson massless⇒long range force
2. A^a_μ carries the symmetry charge (e.g. color —)
3. The quadratic term in $F^{a\mu\nu} \sim \partial A - \partial A + gAA$ is for asymptotic freedom.

Recipe for the construction of theory with local symmetry

1. Write down a Lagrangian with local symmetry
2. Replace the usual derivative $\partial_\mu\phi$ by the covariant derivative $D_\mu\phi \sim \left(\partial_\mu - igA^a_\mu t^a\right)\phi$ where guage fields A^a_μ have been introduced.
3. Use the antisymmetric combination $\left(D_\mu D_\nu - D_\nu D_\mu\right)\phi \sim F^a_{\mu\nu}\phi$ to construct the field tensor $F^a_{\mu\nu}$ and add $-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$ to the Lagrangian density