Spontaneous Symmetry Breaking

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Spontaneous symmetry Breaking

Spontaneous symmetry breaking—-ground state does not have the symmetry of the Hamiltonian
⇒If the symmetry is continuous one, there will be massless scalar fields–Goldstone boson
Example:ferromagnetism
$T > T_c$(Curie temp) all dipoles are randomly oriented–rotational invariant
$T < T_c$ all dipoles are oriented in same direction

Ginzburgh-Landau theory

Free energy as function of magnetization $\vec{M}$ (averaged)

$$\mu(\vec{M}) = (\partial_t \vec{M})^2 + \alpha_1(T)\vec{M}\cdot\vec{M} + \alpha_2(\vec{M}\cdot\vec{M})^2$$

We take $\alpha_2 > 0$ so that the free energy is positive for large M and $\alpha_1(T) = \alpha(T - T_c) \quad \alpha > 0$ so that there is a transition going through Curie temperature T_c. It is easy to see that the ground state is governed by

$$\vec{M}(\alpha_1 + 2\alpha_2\vec{M}\cdot\vec{M}) = 0$$

For $T > T_c$ only solution is $\vec{M} = 0$ and $T < T_c$ non-trivial sol $|\vec{M}| = +\sqrt{\frac{\alpha_1}{2\alpha_2}} \neq 0$

⇒ ground state with $\vec{M}$ in some direction is no longer rotational invariant.

Nambu-Goldstone theorem

Recall that a continuous symmetry will give conserved charge Q. Suppose there are 2 local operators A, B with property

$$[Q, B] = A \qquad Q = \int d^3x\; j_0(x) \quad \textit{indep of time}$$

Suppose $\langle 0|A|0\rangle = v \neq 0$ (symmetry breaking condition)

$$0 \neq \langle 0|[Q, B]|0\rangle = \int d^3x \langle 0|[j_0(x), B]|0\rangle$$

$$= \sum_n (2\pi)^3 \delta^3(\vec{P}_n) \{ \langle 0 | j_0(0) | n \rangle \langle n | B | 0 \rangle e^{-iE_n t} - \langle 0 | B | n \rangle \langle n | j_0(0) | 0 \rangle e^{-iE_n t} \} = v$$

Since $v \neq 0$ and time-independent, we need a state such that

$$E_n \to 0 \qquad for \quad \vec{P}_n = 0$$

massless excitation. For the case of relativistic particle with energy momentum rotation $E = \sqrt{\vec{P}^2 + m^2}$ this implies massless particle- Goldstone boson.

Discrete symmetry case

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \ , \qquad \phi \to -\phi \ \ \textit{symmetry}$$

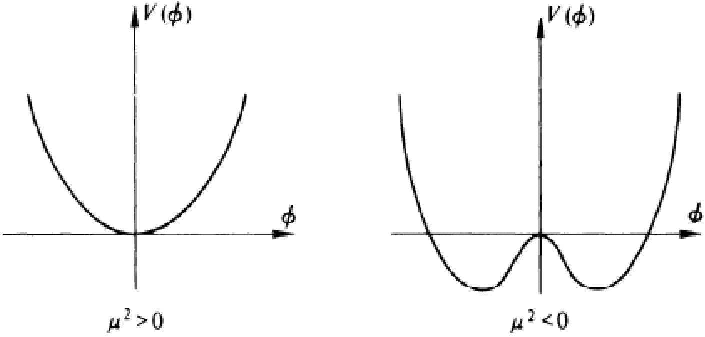
The Hamiltonian density

$$H = \frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

Effective energy

$$\mu(\phi) = \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi) \ \ , \ \ V(\phi) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

For $\mu^2 < 0$ the ground state has $\phi = \pm\sqrt{\frac{-\mu^2}{\lambda}}$ classically.



This means the quantum ground state $|0\rangle$ will have the property

$$\langle 0|\phi|0\rangle = \nu \neq 0 \qquad \text{symmetry breaking condition}$$

Define quantum field ϕ' by $\phi' = \phi - \nu$

$$\textit{then} \quad \mathcal{L} = \frac{1}{2}(\partial_\mu \phi'^2 - (-\mu^2)\phi'^2 - \lambda\nu\phi'^3 - \frac{\lambda}{4}\phi'^4$$

No Goldstone boson—-discrete symmetry

Continuous symmetry case

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2] - V(\sigma^2 + \pi^2)$$

with

$$V(\sigma^2 + \pi^2) = -\frac{\mu^2}{2}(\sigma^2 + \pi^2) + \frac{\lambda}{4}(\sigma^2 + \pi^2)^2$$

This system has O(2) symmetry,

$$\begin{pmatrix} \sigma \\ \pi \end{pmatrix} \rightarrow \begin{pmatrix} \sigma' \\ \pi' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \sigma \\ \pi \end{pmatrix}$$

The minimum is located at

$$\sigma^2 + \pi^2 = \frac{\mu^2}{\lambda} = \nu^2$$

This is a circle *in* $\sigma - \pi$ *plane*. For convenience choose $\langle 0|\sigma|0\rangle = \nu \quad \langle 0|\pi|0\rangle = 0$. New quantum fields are

$$\sigma' = \sigma - \nu \quad , \quad \pi' = \pi$$

The Lagrangian is,

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \sigma'^2 + (\partial_\mu \pi)^2] - \mu^2 \sigma'^2 - \lambda \nu \sigma'(\sigma'^2 + \pi'^2) - \frac{\lambda}{4}(\sigma'^2 + \pi'^2)^2$$

No π'^2 term, $\Rightarrow \pi'$ massless Goldstone boson.
Neother's current is

$$J_\mu(x) = \left[(\partial_\mu \pi)\,\sigma - (\partial_\mu \sigma)\,\pi\right]$$

with associate charge

$$Q = \int d^3 x J_0 = \int d^3 x \left[\left(\partial_0 \pi \right) \sigma - \left(\partial_0 \sigma \right) \pi \right]$$

Using commutation relation we can get,

$$\left[Q, \pi \left(0 \right) \right] = -i \sigma \left(0 \right), \qquad \left[Q, \sigma \left(0 \right) \right] = i \pi \left(0 \right)$$

The vacuum expectation value $\langle 0 | \sigma | 0 \rangle = \nu$ gives the symmetry breaking condition which requires π field to be the massless Goldstone boson. Note that this property is true independent of the perturbation theory.

Higgs Phenomena

When we combine spontaneous symmetry breaking with local symmetry, a very interesting phenomena occurs. This was discovered in the $60's$ by Higgs, Englert & Brout, Guralnik, Hagen & Kibble independently.

Abelian case

Consider the Lagrangian given by

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) + \mu^2 \phi \phi^\dagger - \lambda(\phi^\dagger \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where

$$D^\mu \phi = (\partial^\mu - igA^\mu)\phi \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The Lagrangian is invariant under the local gauge transformation

$$\phi(x) \to \phi' = e^{-i\alpha(x)} \phi(x)$$

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \frac{1}{g} \partial_\mu \alpha(x)$$

The spontaneous symm. breaking is generated by the potential

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2$$

which has a minimum at

$$\phi^\dagger \phi = \frac{\nu^2}{2} = \frac{1}{2}(\frac{\mu^2}{\lambda})$$

For the quantum theory, we can choose

$$|\langle 0|\phi|0\rangle| = \frac{\nu}{\sqrt{2}}$$

Or if we write

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

With the choice

$$\langle\phi_1\rangle = v \quad , \quad \langle\phi_2\rangle = 0$$

ϕ_2 corresponds to Goldstone boson as before. Define the quantum fields by

$$\phi_1' = \phi_1 - v \quad , \quad \phi_2' = \phi_2$$

Covariant derivative terms gives

$$(D_\mu\phi)^+(D^\mu\phi) = [(\partial_\mu + igA_\mu)\phi^+][(\partial^\mu - igA^\mu)\phi]$$

$$\frac{-1}{2}(\partial_\mu\phi_1' + gA_\mu\phi_2')^2 + \frac{1}{2}(\partial_\mu\phi_2' - gA_\mu\phi_1')^2 + \underline{\frac{g^2v^2}{2}A^\mu A_\mu} + \cdots \text{ mass terms for } A^\mu$$

Write the scalar field as

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x))e^{i\xi(x)/v}$$

"Gauge" transformation:

$$\phi \longrightarrow \phi' = e^{-i\xi(x)/v}\phi(x) \quad , \quad B_\mu = A_\mu(x) - \frac{1}{gv}\partial_\mu\xi$$

$\xi(x)$ disappears from the Lagrangian
Thus massless gauge field A_μ combine with Goldstone boson $\xi(x)$ to become massive gauge boson. As a consequence, two long range forces (from Goldstone boson $\xi(x)$ and $A_\mu(x)$) disappear.

Non-Abelian case

SU(2) group: $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ doublet

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$V(\phi) = -\mu^2 (\phi^\dagger \phi) + \lambda (\phi^\dagger \phi)^2$$

Spontaneous symmetry breaking:
Minimum

$$\frac{\partial V}{\partial \phi_i} = \left[-\mu^2 + 2(\phi^\dagger \phi) \right] \phi_i = 0$$

$\Longrightarrow$

$$-\mu^2 + 2(\phi^\dagger \phi) = 0$$

Simple choice

$$\langle \phi \rangle_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \nu \end{pmatrix} \qquad \nu = \sqrt{\frac{\mu^2}{\lambda}}$$

Define

$$\phi' = \phi - \langle \phi \rangle_0$$

From covariant derivative

$$(D_\mu \phi)^\dagger (D^\mu \phi) = [\partial_\mu - ig \frac{\vec{\tau} \cdot \vec{A}_\mu}{2} (\phi' + \langle \phi \rangle_0)]^\dagger [\partial^\mu - ig \frac{\vec{\tau} \cdot \vec{A}_\mu}{2} (\phi' + \langle \phi \rangle_0)]$$

$$\rightarrow \frac{1}{4} g^2 \langle \phi \rangle_0 (\vec{\tau} \cdot \vec{A}_\mu)(\vec{\tau} \cdot \vec{A}^\mu) \langle \phi \rangle_0 = \frac{1}{2} (\frac{gv}{2})^2 \vec{A}_\mu \cdot \vec{A}^\mu$$

All gauge bosons get masses

$$M_A = \frac{1}{2} g v$$

The symmetry is completely broken. Write

$$\phi(x) = \exp\{\frac{i\vec{\tau}\cdot\vec{\xi}(x)}{v}\} \left(\begin{array}{c} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{array} \right)$$

Use "gauge" transformation

$$\phi'(x) = U(x)\phi(x) = \frac{1}{\sqrt{2}} \left(\begin{array}{c} 0 \\ v+\eta(x) \end{array} \right)$$

$$\frac{\vec{\tau}\cdot\vec{B}_\mu}{2} = U\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}U^{-1} - \frac{i}{g}[\partial_\mu U]U^{-1}$$

$$\text{where} \quad U(x) = \exp\{\frac{\vec{\tau}\cdot\vec{\xi}}{v}\}$$

to transform away $\vec{\xi}(x)$.