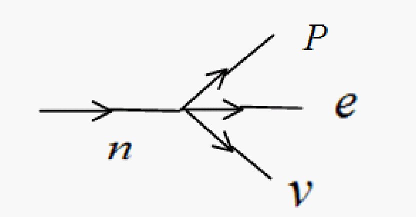
Standard Model

Ling-Fong Li

Milestones of Weak Interaction

1. **Neutrino and Nuclear β decay**,
   The e^- from nuclei decay,
   $$(A, Z) \to (A, Z+1) + e^-$$
   have continuous energy spectrum. If basic mechanism were
   $$n \longrightarrow p + e^-$$
   the energy momentum conservation will require e^- to have a single energy. Pauli (1930) postulated the presence of **neutrino** which carries away energy and momentum,
   $$n \longrightarrow p + e^- + \bar{\nu}_e$$

2. **Fermi Theory**
   Fermi (1934) proposed to write weak interaction in the form,
   $$\mathcal{L}_F = \frac{G_F}{\sqrt{2}}[\bar{p}(x)\gamma_\mu n(x)][\bar{e}(x)\gamma^\mu \nu(x)] + h.c. \quad G_F : \text{ Fermi coupling constant}$$
   Fitting nuclear β decay reates give
   $$G_F \simeq \frac{10^{-5}}{M_p^2}, \qquad M_p \text{ proton mass}$$
   This works very well for $\Delta J = 0$, β-decays of many nuclei.



Later, Gamow-Teller interaction was added

$$\mathcal{L}_{GT} = \frac{G_F}{\sqrt{2}}[\bar{p}(x)\gamma_\mu\gamma_5 n(x)][\bar{e}(x)\gamma^\mu\gamma_5\nu_e(x)] + h.c.$$

to account for $\Delta J = 1$ nuclear β decays.

3. Parity violation and V - A theory
$\theta - \tau$ puzzle
In 1950's, two decays were observed,

$$\theta \to \pi^+ + \pi^-, \qquad \text{(even parity)}$$

$$\tau \to \pi^+ + \pi^- + \pi^0, \qquad \text{(odd parity)}$$

while θ and τ have same mass, charge and spin. Hard to understand these if the parity is a good symmetry.

1956 : Lee and Yang proposed that parity is not conserved.
1957 : C. S. Wu showed that e^- in ^{60}Co decay has the property,

$$\left\langle \vec{\sigma} \cdot \vec{p} \right\rangle \neq 0, \qquad \vec{\sigma},\ \vec{p} \text{ spin and momentum of } e^-$$

This implies parity violation.
V-A theory (1958 Feynman and Gell-Mann, Sudarshan and Marshak, Sakurai)
As a result of parity violation, weak interaction was written with $V - A$ currents,

$$L_{eff} = \frac{G_F}{\sqrt{2}} J_\mu^\dagger J^\mu + h.c.$$

where

$$J_\lambda(x) = J_{l\,\lambda}(x) + J_{h\,\lambda}(x)$$

$$J_l^\lambda(x) = \bar{\nu}_e \gamma^\lambda (1 - \gamma_5) e + \bar{\nu}_\mu \gamma^\lambda (1 - \gamma_5) \mu, \qquad \text{leptonic current} \qquad (1)$$

and

$$J_h^\lambda(x) = \bar{u} \gamma^\lambda (1 - \gamma_5)(\cos\theta_c d + \sin\theta_c s) \qquad \text{hadronic current}$$

$$\theta_c \ : \ \text{Cabibbo angle}$$

Note that in V-A form, fermions are all left-handed.
Define

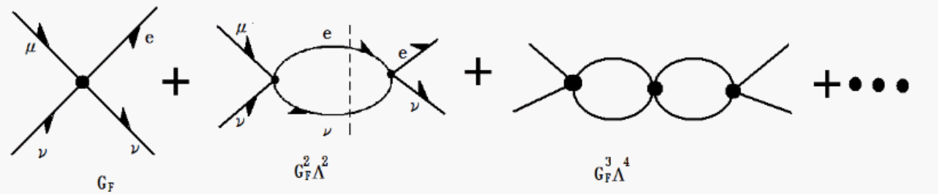
$$\psi_L \equiv \frac{1}{2}(1 - \gamma_5)\psi$$

Then we can simplify the weak currents,

$$J_l^\lambda(x) = 2\bar{\nu}_{eL} \gamma^\lambda e_L + 2\bar{\nu}_{\mu L} \gamma^\lambda \mu_L + \ldots$$

Difficulties:
(1) **Not renormalizable**

In Fermi theory, 4 fermions interaction has dimension 6 and is not renormalizable. The higher order graphs are more and more divergent. For example, in μ decay,



(2) **Violate unitarity**
The tree amplitude for $\nu_\mu + e \to \mu + \nu_e$ has only J = 1 partial wave at high energies and cross section has the form,

$$\sigma(\nu_\mu e) \approx G_F^2 S, \qquad S = 2m_e E$$

On the other hand, unitarity for J=1 cross section is

$$\sigma(J=1) < \frac{1}{S}$$

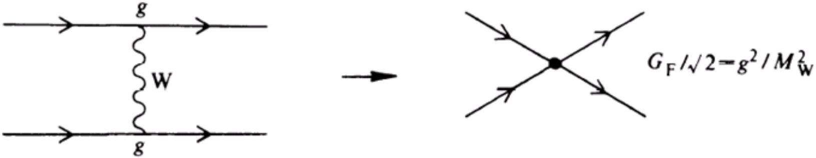
Thus $\sigma(\nu_\mu e)$ violates unitarity for $E \geq 300\ GeV$. Since unitarity comes from conservation of probablity, this violation is unacceptable.

Intermidate Boson Theory(IVB)

In analogy with QED, introduce W to couple to the V-A current

$$\mathcal{L}_W = g(J_\mu W^\mu + h.c.)$$

For example, the μ decay is mediated by W-exchange.



Since weak interaction is short range, $M_W \neq 0$. From W-boson propagator

$$\frac{-g^{\mu\nu} + \frac{k^\mu k^\nu}{M_W^2}}{k^2 - M_W^2} \rightarrow \frac{g^{\mu\nu}}{M_W^2} \quad \text{when} \quad |k_\mu| \ll M_W$$

This reproduces 4-fermion interaction with $\dfrac{g^2}{M_W^2} = \dfrac{G_F}{\sqrt{2}}$

In this theory, $\nu_\mu + e \to \mu + \nu_e$ no longer violates unitarity. But the violation of unitarity shows up in

$$\nu + \bar{\nu} \to W^+ + W^-$$

and the theory is still non-renormalizable.

Construction of SU(2)×U(1) model

Idea : combine local symmetry with symmetry breaking

Choice of group

In IVB theory,

$$\mathcal{L}_W = g(J_\mu W^\mu + h.c)$$

For simplicity neglect all other fermions except ν, e

$$J_\mu = \bar{\nu}\gamma_\mu(1-\gamma_5)e$$

In electromagnetic interaction, we have

$$\mathcal{L}_{em} = eJ_\mu^{em}A^\mu, \qquad \text{where} \qquad J_\mu^{em} = \bar{e}\gamma_\mu e$$

Define electromagnetic and weak charges as the intergals

$$T_+ = \frac{1}{2}\int d^3x J_0(x) = \frac{1}{2}\int d^3x\ \nu^\dagger(1-\gamma_5)e, \qquad T_- = (T_+)^\dagger$$

$$Q = \int d^3x J_0^{em}(x) = -\int d^3x e^\dagger e$$

Compute the commutator $[T_+, T_-] = 2T_3$ and

$$T_3 = \frac{1}{4}\int d^3x[\nu^\dagger(1-\gamma_5)\nu - e^\dagger(1-\gamma_5)e] \neq Q$$

These 3 charges, T_+, T_- and Q don't form a $SU(2)$ algebra. Note weak charges $T_\pm$ have $V - A$ form while the em charge Q is pure vector.
At this point, there are 2 alternatives:

1. Introduce another guage boson coupled to T_3. This leads to group $SU(2) \times U(1)$. This is the choice we will adapt eventually.
2. Add new fermions such that T_+, T_- and Q do form a SU(2) algebra (Georgi and Glashow 1972) e.g.

$$\frac{1}{2}(1-\gamma_5)\begin{pmatrix} E^+ \\ \nu_e\cos\alpha + N\sin\alpha \\ e^- \end{pmatrix}$$

$$\frac{1}{2}(1+\gamma_5)\begin{pmatrix} E^+ \\ N \\ e^- \end{pmatrix}$$

and a singlet

$$\frac{1}{2}(1+\gamma_5)(N\cos\alpha - \nu_e\sin\alpha)$$

so that weak charge is

$$T_+ = \frac{1}{2}\int d^3x \left[E^+ (1-\gamma_5)(\nu_e \cos\alpha + N \sin\alpha)\right]$$
$$+ (\nu_e \cos\alpha + N \sin\alpha)(1-\gamma_5)\, e + E^+ (1+\gamma_5)\, N + N^\dagger (1+\gamma_5)\, e$$

We can verify that

$$[T_+, T_-] = 2Q$$

with

$$Q = \int d^3x \left[E^\dagger E - e^\dagger e\right]$$

Clearly, here only electromagnetic current is neutral and is ruled out by the discoveries of neutral weak current reactions in 1973.

Now choose gauge group to be $SU(2) \times U(1)$. The Lagrangian for the gauge fields is

$$L = -\frac{1}{4} F^{i\mu\nu} F^{i}_{\mu\nu} - \frac{1}{4} G^{\mu\nu} G_{\mu\nu}$$

where

$$F^{i}_{\mu\nu} = \partial_\mu A^{i}_{\nu} - \partial_\nu A^{i}_{\nu} + g\epsilon^{ijk} A^{j}_{\mu} A^{k}_{\nu} \qquad SU(2) \quad \text{gauge fields}$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\mu B_\mu \qquad U(1) \quad \text{gauge field}$$

Fermions

Clearly, from structure of weak charged current given in Eq(1) ν, e form a doublet under SU(2),

$$l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$$

Then

$$T_+ = \int (\nu_L^+ e_L) d^3x, \quad T_- = \int (e_L^+ \nu_L) d^3x, \quad Q = \int (e_L^+ e_L + e_R^+ e_R)$$

Note that

$$Q - T_3 = \int [-\frac{1}{2}(\nu_L^+ \nu_L + e_L^+ e_L) - e_R^+ e_R] d^3x$$

We can show that

$$[Q - T_3, T_i] = 0\ , \quad i = 1, 2, 3$$

Take $Q - T_3$ to be $U(1)$ charge $Y \equiv 2(Q - T_3)$, called **weak hypercharge**. The Y charges for fermions are

$$l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \qquad Y = -1, \qquad e_R \;\; Y = -2$$

Lagrangian for gauge coupling is

$$\mathcal{L}_2 = \bar{l}_L i\gamma^\nu D_\nu l_l + \bar{l}_R i\gamma^\nu D_\nu l_R \tag{2}$$

where

$$D_\nu \psi = (\partial_\nu - ig\frac{\vec{\tau}\cdot\vec{A}_\nu}{2} - ig'\frac{Y}{2}B_\nu)\psi$$

For example,

$$D_\nu l_L = (\partial_\nu - ig\frac{\vec{\tau}\cdot\vec{A}_\nu}{2} - ig'\frac{Y}{2}B_\nu) l_L$$

Spontaneous Symmetry Breaking

Symmetry braking pattern we want is $SU(2) \times U(1) \rightarrow U(1)_{em}$. Choose scalar fields in SU(2) doublet with hypercharge $Y = 1$,

$$\phi = \begin{pmatrix} \phi^{\dagger} \\ \phi^{0} \end{pmatrix}, \; Y = 1$$

Lagrangian contaning ϕ is,

$$\mathcal{L}_3 = (D_\mu \phi)^{\dagger}(D^{\mu}\phi) - V(\phi)$$

where

$$D_\mu \phi = (\partial_\mu - \frac{ig}{2}\vec{\tau} \cdot \vec{A}_\mu - \frac{ig^{'}}{2} B_\mu)\phi$$

and

$$V(\phi) = -\mu^2 \phi^{\dagger}\phi + \lambda(\phi^{\dagger}\phi)^2$$

Coupling between leptons and scalar field ϕ,

$$\mathcal{L}_4 = f \bar{L}_L \phi e_R + h.c.$$

Spontaneous symmetry breaking is generated by the vaccum expectation value

$$< \phi >_0 = \langle 0 | \, \phi \, | 0 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \qquad \text{with} \quad v = \sqrt{\frac{\mu^2}{\lambda}}$$

Write the scalar field in the form

$$\phi(x) = U^{-1}(\vec{\xi}) \left(\begin{array}{c} 0 \\ \frac{v+\eta(x)}{\sqrt{2}} \end{array} \right) \quad \text{where} \quad U(\vec{\xi}) = \exp[\frac{i\vec{\xi}(x)\cdot\vec{\tau}}{v}] \tag{3}$$

Gauge Transformation

We can then simplify the form of scalar by a gauge transformation

$$\phi' = U(\vec{\xi})\phi = \frac{1}{\sqrt{2}} \left(\begin{array}{c} 0 \\ v+\eta(x) \end{array} \right)$$

$$\frac{\vec{\tau}\cdot\vec{A}'_\mu}{2} = U(\vec{\xi})\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}U^{-1}(\vec{\xi}) - \frac{i}{g}(\partial_\mu U)U^{-1}$$

field $\vec{\xi}(x)$ disappears from Lagrangian because of gauge invariance. From $\mathcal{L}_4$ (Yukawa coupling), VEV of the scalar field gives

$$\mathcal{L}_4 = f\frac{1}{\sqrt{2}}(\bar{l}_L < \phi > e_R + h.c.) + f\frac{\eta(x)}{\sqrt{2}}(\bar{e}_L e_R + h.c.)$$

the electron is now massive withs

$$m_e = \frac{f}{\sqrt{2}}v$$

Mass spectrum

We now list the mass spectrum of the theory after the spontaneous symmetry breaking:

1. Fermion mass
$$m_e = \frac{fv}{\sqrt{2}}$$
2. Scalar mass(Higgs)
$$V(\phi^{'}) = \mu^2\eta^2 + \lambda v \eta^3 + \frac{\lambda}{4}\eta^4 \qquad \longrightarrow \quad m_\eta = \sqrt{2}\mu$$
3. Gauge boson masses
From covariant derivative in $\mathcal{L}_3$
$$\mathcal{L}_3 = \frac{v^2}{2}\chi^\dagger(g\frac{\vec{\tau}\cdot\vec{A}_\mu^{'}}{2} + \frac{g^{'}B_\mu^{'}}{2})(g\frac{\vec{\tau}\cdot\vec{A}^{'\mu}}{2} + \frac{g^{'}B^{'\mu}}{2})\chi + \cdots, \qquad \chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

we get the mass terms for the gauge bosons,
$$\begin{aligned} \mathcal{L}_3 &= \frac{v^2}{8}\{g^2[(A_\mu^{\mathbf{1}})^2 + (A_\mu^{\mathbf{2}})^2] + (gA_\mu^{\mathbf{3}} - g^{'}B_\mu)^2\} + \cdots \\ &= M_W^2 W^{+\mu}W_\mu^- + \frac{1}{2}M_Z^2 Z^\mu Z_\mu + \cdots \end{aligned}$$

where
$$W_\mu^+ = \frac{1}{\sqrt{2}}(A_\mu^{\mathbf{1}} - iA_\mu^{\mathbf{2}}), \qquad M_W^2 = \frac{g^2v^2}{4}$$

$$Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(g' A^3 - g B_\mu), \quad M_Z^2 = \frac{g^2 + g'^2}{4} v^2$$

The field

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(g' A_\mu^3 + g B_\mu)$$

is massless photon.
For convience we define

$$\tan\theta_W = \frac{g'}{g} \quad \theta_W : \text{Weinberg angle or weak mixing angle}$$

Then we can write

$$Z_\mu = \cos\theta_W A_\mu^3 - \sin\theta_W B_\mu \quad M_Z^2 = \frac{g^2 v^2}{4}\sec^2\theta_W$$

$$A_\mu = \sin\theta_W A_\mu^3 - \cos\theta_W B_\mu$$

Note that there is a relation of the form,

$$\rho = \frac{M_W^2}{M_Z^2\cos^2\theta_W} = 1$$

which is a consequence of the doublet nature of the scalar fields.

The weak interactions mediated by W and Z bosons can be read out from Eq(2)

1. Charged current

$$\mathcal{L}_{cc} = \frac{g}{\sqrt{2}}(J^\dagger_\mu W^{\dagger\mu} + h.c.) \quad J^\dagger_\mu = J^1_\mu + iJ^2_\mu = \frac{1}{2}\bar{\nu}\gamma_\mu(1-\gamma_5)e$$

Again to get 4-fermion interaction as low energy limit, we require

$$\frac{g^2}{8M_W^2} = \frac{G_F}{\sqrt{2}}$$

which implies that

$$v = \sqrt{\frac{\sqrt{2}}{G_F}} \approx 246\,Gev$$

This is usually referred to as the **weak scale**.

2. Neutral Current

The Largrangian for the neutral currents is

$$\mathcal{L}_{NC} = gJ^3_\mu A^{\mathbf{3}\mu} + \frac{g^{'}}{2} J^Y_\mu B^\mu = eJ^{em}_\mu A^\mu + \frac{g}{\cos\theta_W} J^Z_\mu Z^\mu$$

where

$$e = g\sin\theta_W,$$

and

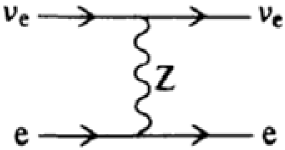
$$J^Z_\mu = J^{\mathbf{3}}_\mu - \sin^2\theta_W J^{em}_\mu$$

is the weak neutral current. We can define the weak neutral charge as

$$Q^Z = \int J^Z_0 d^3x = (T_3 - \sin^2\theta_W Q)$$

So the coupling strength of fermions to Z-boson is proportional to $T_3 - \sin^2\theta_W Q$. In particluar, Z boson can contribute to the scattering

$$\nu_e + e \longrightarrow \nu_e + e$$



The measurement of this corss section in the 1970's give $\sin^2\theta_W \approx 0.22$. This yields $M_W \approx 80$ GeV and $M_Z \approx 90$ GeV.

Generalization to more than one family.
From 4-fermion and IVB theory, the form of weak currents of leptons and hadrons gives the following multiplets structure,

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \quad e_R, \mu_R \quad \begin{pmatrix} u \\ d_\theta \end{pmatrix}_L \quad u_R, d_R, s_R$$

where

$$d_\theta = \cos\theta_C d + \sin\theta_C s$$

The neutral current in the down quark sector is

$$\begin{aligned} \mathcal{L}_{NC} &= [\bar{d}_\theta \gamma_\mu (-\frac{1}{2} + \sin^2\theta_C \frac{1}{3}) d_{\theta L} - \sin^2\theta_W \frac{1}{3} (\bar{d}_R \gamma_\mu d_R + \bar{s}_R \gamma_\mu s_R)] Z^\mu \\ &= (-\frac{1}{2} + \sin^2\theta_W \frac{1}{3}) [(\bar{d}_L \gamma_\mu d_L + \bar{s}_L \gamma_\mu s_L) + \sin\theta_W \cos\theta_W (\bar{d}_L \gamma_\mu s_L + \bar{s}_L \gamma_\mu d_L) + \ldots \end{aligned}$$

The term $\left(\bar{d}_L \gamma_\mu s_L + \bar{s}_L \gamma_\mu d_L\right)$ gives rise to $\Delta S = 1$ neutral current processes, e.g. $K_L \to \mu^+ + \mu^-$ with same order of magnitude as charged current interaction. But experimentally,

$$R = \frac{\Gamma(K_L \to \mu^+ + \mu^-)}{\Gamma(K^+ \to \mu + \nu)} \leq 10^{-8}$$

Thus we can not have $\Delta S = 1$ neutral curent process at the same order of magnitude as the charged current process.
GIM mechanism

Glashow, Iliopoulos and Maiani (1970) suggested a 4-th quark, the charm quark c, which couples to the orthogonal combination $s_\theta = -\sin\theta_c d + \cos\theta_c s$ so that

$$\begin{pmatrix} u \\ d_\theta \end{pmatrix}_L , \quad \begin{pmatrix} c \\ s_\theta \end{pmatrix}_L$$

As a result , the $\Delta S = 1$, neutral current is canceled out. The new current is of the form

$$\bar{d}_\theta(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_W)\gamma_\mu d_\theta + \bar{s}_\theta(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_W)\gamma_\mu s_\theta = (-\frac{1}{2}+\frac{1}{3}\sin^2\theta_W)(\bar{d}\gamma_\mu d + \bar{s}\gamma_\mu s)$$

which conserves the strangeness. This avoids the conflict with exp on $K_L \longrightarrow \mu^+\mu^-$

Quark mixing

Before spontaneous symmetry braking, fermions are all massless becauseψ_L and ψ_R have different quantum numbers under $SU(2)\times U(1)$ i.e. mass term $(\bar{\psi}_L\psi_R + h.c.)$ is not invariant under $SU(2)\times U(1)$. For more than one doublets, ψ_{iR}, ψ_{iL} have same quantum numbers under $SU(2)\times U(1)$ group we call "weak eigenstates". After spontaneous symmetry breaking, fermions obtain their masses through Yukawa coupling.

$$\mathcal{L}_Y = (f_{ij}\bar{g}_{iL}u_{Rj} + f'_{ij}\bar{g}_{iL}d_{Rj})\phi + h.c.$$

Renormalizability requires all possible terms consistent with $SU(2)\times U(1)$ symmetry. Since f_{ij},, f'_{ij} are arbitrary, fermion mass matrices are not diagonal.

$$\text{mass eigenstates} \neq \text{weak eigenstates}$$

The mass matrices in up and down sectors are

$$m_{ij}^{(u)} = f_{ij}\frac{v}{\sqrt{2}} \quad m_{ij}^{(d)} = f'_{ij}\frac{v}{\sqrt{2}}$$

These matrices which are sandwiched between left and right handed fields can be diagonalized by bi-unitary transformations, i.e. given a mass matrix m_{ij},there exits unitary matrices S and T such that

$$S^{\dagger} m T = m_d$$

is diagonal. S is the unitary matrix which diagnoalizes the hermitian combination mm^{+}, i. e.

$$S^{\dagger}(mm^{\dagger})S = m_d^2$$

<u>Biunitary transformation</u>
Write

$$m_d^2 = \begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix}$$

Define

$$m_d = \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix}$$

and

$$H = S m_d S^{\dagger} \qquad \text{hermitian}$$

Define a matrix V by

$$V \equiv H^{-1} m$$

Then

$$VV^{\dagger} = H^{-1} m m^{\dagger} H^{-1} = H^{-1} S m_d^2 S^{\dagger} H^{-1} = H^{-1} H^2 H^{-1} = 1$$

So V is unitary and we have

$$S^{\dagger} H S = m_d, \qquad \Longrightarrow \qquad S^{\dagger} m V^{\dagger} S = m_d$$

Or

$$S^{\dagger} m T = m_d, \qquad \text{with} \qquad T = V^{\dagger} S$$

————————-

If we write the doublets, (weak eigenstates) as

$$q_{1L} = \begin{pmatrix} u' \\ d' \end{pmatrix}_L \quad q_{2L} = \begin{pmatrix} c' \\ s' \end{pmatrix}_L$$

These weak eigenstates are related to mass eigenstates by unitary transformations,

$$\begin{pmatrix} u' \\ c' \end{pmatrix} = S_u \begin{pmatrix} u \\ c \end{pmatrix}, \qquad \begin{pmatrix} d' \\ s' \end{pmatrix} = S_d \begin{pmatrix} d \\ s \end{pmatrix}$$

Note that in the coupling to charged gauge boson $W^{\pm}$, we have

$$\mathcal{L}_W = \frac{g}{\sqrt{2}} W_\mu [\bar{q}_{1L}\gamma^\mu \tau^\dagger q_{1L} + \bar{q}_{2L}\gamma^\mu \tau^\dagger q_{2L}] + h.c.$$

and is invariant under unitary transformation in q_{1L}, q_{2L} space, i.e.

$$\begin{pmatrix} q'_{1L} \\ q'_{2L} \end{pmatrix} = V \begin{pmatrix} q_{1L} \\ q_{2L} \end{pmatrix} \quad VV^\dagger = 1 = V^\dagger V$$

We can use this feature to put all mixing in the down quark sector,

$$q'_{iL} = \begin{pmatrix} u \\ d'' \end{pmatrix}_L, \begin{pmatrix} c \\ s'' \end{pmatrix}_L, \qquad \text{where} \quad \begin{pmatrix} d'' \\ s'' \end{pmatrix} = U \begin{pmatrix} d \\ s \end{pmatrix}$$

Here U is a 2×2 unitary matrix. Clearly, we can extend this to 3 generations with result

$$q_{iL} : \begin{pmatrix} u \\ d'' \end{pmatrix}, \begin{pmatrix} c \\ s'' \end{pmatrix}, \begin{pmatrix} t \\ b'' \end{pmatrix}, \qquad \begin{pmatrix} d'' \\ s'' \\ b'' \end{pmatrix} = U \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

Now U is a 3×3 unitary matrix, usually called the Cabibbo-Kobayahsi-Maskawa (CKM) matrix.

CP violation Phase

CP violetion can come form complex coupling to gauge bosons. The coupling of $W^{\pm}$ to quarks is governed by the 3×3 unitary matrix U . This unitary matrix U can have many complex entries. However, in diagonalzing mass matrices, $S^{\dagger}(mm^{\dagger})S = m_d^2$. There is arbitrariness in the matrix S, in the form of diagonal phases i.e. if S diagonalizes the mass matrix, so does $S^{'}$

$$S^{'} = S \begin{pmatrix} e^{i\alpha_1} & \dots & \dots \\ \vdots & \ddots & \vdots \\ \vdots & \dots & e^{\alpha_n} \end{pmatrix}$$

We can then redefine the quark fields to get rid of some phases in U. It turns out that for n$\times$n unitary matrix, number of independent physical phases left over is

$$\frac{(n-1)(n-2)}{2}$$

Thus to get CP violetion we need 3 generations or more (Kobayashi Mskawa). Here we give a constructive proof of this statement. Start with a first doublet written as,

$$q_{1L} = \begin{pmatrix} u \\ U_{11}d + U_{12}s + U_{13}b \end{pmatrix}$$

If U_{11} has phase δ,

$$U_{11} = R_{11}e^{i\delta}, \qquad R_{11} \qquad \text{real}$$

then δ can be absorbed in the redefiniton of the $u-$quark

$$u \longrightarrow u' = ue^{-i\delta}$$

and we can write

$$q_{1L} = e^{i\delta} \begin{pmatrix} u' \\ R_{11}d + U'_{12}s + U'_{13}b \end{pmatrix}$$

Similarly, we can factor out the complex phases of U_{21} and U_{31} by redefinition of c and t quark fields. These overall phases are immaterial. Finally we can absorb two more phases of U_{12} and U_{13} by a redefinition of the s and b fields. The doublets now take the form

$$\begin{pmatrix} u' \\ R_{11}d + R_{12}s + R_{13}b \end{pmatrix}_L, \qquad \begin{pmatrix} c' \\ R_{21}d + R_{22}e^{i\delta_1}s + R_{23}e^{i\delta_2}b \end{pmatrix}_L,$$

$$\begin{pmatrix} t' \\ R_{31}d + R_{32}e^{i\delta_3}s + R_{33}e^{i\delta_4}b \end{pmatrix}_L,$$

Now we have reduced the number of parameters to 13. The normalization conditons of each down-like state gives 3 real conditions and orthogonality conditions among different states give 6 real conditions on the parameters, Now we are down to 4 parameters. Since we need 3 parameters for the real orthogonal matrix, we end up with one independent phase.

Flavor consevation in neutral current interaction

The coupling of neutral Z boson to the fermions conserve flavors. This can be illustrated as follows. Write the neutral currents in terms of weak eigenstates,

$$J_{\mu}^{Z} = \sum_{i} \bar{\psi}_{i} \gamma_{\mu} \left[T_{3} \left(\psi_{i} \right) - \sin^{2} \theta_{W} Q \left(\psi_{i} \right) \right] \psi_{i}$$

Separate into left- and right-handed fields and distingush the up and down components,

$$\begin{aligned} J_{\mu}^{Z} \quad = \quad & \sum_{i} (\bar{u}'_{Li} \gamma_{\mu} \left[\frac{1}{2} - \sin^{2} \theta_{W} \left(\frac{2}{3} \right) \right] u'_{Li} + \bar{d}'_{Li} \gamma_{\mu} \left[-\frac{1}{2} + \sin^{2} \theta_{W} \left(\frac{1}{3} \right) \right] d'_{Li} \\ & + \bar{u}'_{Ri} \gamma_{\mu} \left[- \sin^{2} \theta_{W} \left(\frac{2}{3} \right) \right] u'_{Ri} + \bar{d}'_{Ri} \gamma_{\mu} \left[\sin^{2} \theta_{W} \left(\frac{1}{3} \right) \right] d'_{Ri} \end{aligned}$$

Since weak eigen states q_{iL} and mass eigen states q'_{iL} are related by unitary matries,

$$u'_{Li} = U \left(u_{L} \right)_{ij} u_{Lj}, \qquad \cdots$$

We see tha unitary matrices cancel out in the combination, $\bar{u}'_{Li} u'_{Li}$ so that the neutral current in terms of mass eigenstates has the same form as the one in terms of weak eigenstates. Thus it conserves all quark flavor. Note this feature is due to the fact that all quarks with same helicity and electric charge have the same quantum number with repect to $SU(2) \times U(1)$ gauge group.